VISCOSITY METHOD FOR THE DESCRIBING OF DYNAMICAL SYSTEMS

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Three types of nonlinear partial differential equations of polynomial form are considered and explicit substitutions of dependent variables, which transform the equations under study to linear equations, are obtained. Some nonlinear second order partial differential equations, which can be solved by the viscosity method, are also obtained

Introduction

The problem of finding exactly solvable nonlinear partial differential equations has very popular after discovering the inverse scattering method [1] and it still attracts the attention of many authors[2]. There are also some other methods of obtaining exact solutions of nonlinear equations: see [3-6]. However, in spite of variety of forms, all methods know till now are based after all on the very simple idea-to reduce the problem of solving the given nonlinear equation by means of some transformations to the problem of solving a more simple equation, which has been already studied. In the most of cases, people try to reduce nonlinear problems to linear ones because the theory of linear equations is well elaborated, although sometimes the reduction to a more simple nonlinear equation is also used.

Therefore, the following important and interesting problem arises: to describe the class of nonlinear equations which can be reduced by means of some transformations to linear equations.

One of possible ways of solving this problem is to apply all kinds of transformations to a given class of linear equations and to analyze the nonlinear equations thus obtained [7-8]. Of course, this class of equations is very small in comparison with the class of all exactly solvable equations. Nonetheless, we know this small class contains some physically interesting equations, for example, the Burgers-Hopf equation [9]. We hope that the methodical study of all possible substitutions of variables and the initial linear equations may lead to some new exactly solvable nonlinear equations being of physical interest.

In this paper, we investigate some nonlinear equations obtained from linear partial differential equations of the second order by substitutions of dependent variables.

Nonlinear forms of linear equations

Since equations occuring in physical applications contain the derivatives with respect to the time variable *t*, as a rule, not higher than of the second order, we consider the nonlinear forms of the following general linear partial differential equation of the second order

$$a\psi_t+b\ \psi_{kt}+c\psi_{tt}+d\psi_{xx}=0 \quad (1)$$

where *a*, *b*, *c* and *d* are arbitrary functions of variables *x* and *t*.

Making the substitution

$$\psi=exp(\alpha\varphi+\beta\varphi_x+\gamma\varphi_t) \quad (2)$$

we obtain the following equation (we confine ourselves to the equations of polynomial type):

$$\begin{aligned} &a(\alpha\varphi_t+\beta\varphi_{xt}+\gamma\varphi_{tt})+b(\alpha^2\varphi_t\cdot\varphi_x+\alpha\gamma\varphi_{tt}\varphi_x+\alpha\beta\varphi_t\varphi_{xx}+\beta^2\varphi_{xt}+\beta\gamma\varphi_{tt}\cdot\varphi_{xx}+\gamma^2\varphi_{tt}\varphi_{tx}+\alpha\gamma\varphi_t\varphi_{xt}+\\ &+\beta\gamma\varphi_{xt}+\alpha\varphi_{xt}+\beta\varphi_{xt}+\gamma\varphi x_{tt})+c(\alpha^2\varphi_t^2+\beta^2\varphi_{xt}^2+\gamma^2\varphi_{tt}^2+2\alpha\beta\varphi_{xt}\varphi_t+2\alpha\gamma\varphi_{tt}\varphi_t+2\beta\gamma\varphi x_t\varphi_{tt}+\alpha\varphi_{tt}+\\ &+\beta\varphi_{xtt}+\gamma\varphi_{ttt})+d(\alpha^2\varphi_x^2+\beta^2\varphi_{xx}^2+\gamma^2\cdot\varphi_{tx}^2+2\alpha\beta\varphi_{xx}\cdot\varphi_x+2\alpha\gamma\varphi_{tx}\cdot\varphi_x+2\beta\gamma\varphi_{xx}\cdot\varphi_{tx}+\alpha\varphi_{xx}+\beta\varphi_{xxx}+\gamma\varphi_{txx})=0 \end{aligned} \quad (3)$$

Equation (3) is linear with respect to the derivatives of the third order and nonlinear with respect to the derivatives of the second and first order. One can check that equations of this kind can be reduced to three different types of equations by means of linear replacement of independent variables:

$$\varphi_{xxx}+L(\varphi_{xx},\varphi_{tt},\varphi_{xt},\varphi_x,\varphi_t\ldots)=0 \quad (4)$$

$$\varphi_{xxt}+M(\varphi_{xx},\varphi_{tt},\varphi_{xt},\varphi_x,\varphi_t\ldots)=0 \quad (5)$$

$$\varphi_{xxx}+\varphi_{xtt}+N(\varphi_{xx},\varphi_{tt},\varphi_{xt},\varphi_x,\varphi_t\ldots)=0 \quad (6)$$

(and symmetrically $x\leftrightarrow t$)

It is not difficult to show that the equation of type (4) is obtained from eq.(3) provided

$$b=c=\gamma=0 \quad (7)$$

that is, we have the following equation:

$$a(\alpha\varphi_t+\beta\varphi_{xt})+d\left(\alpha^2\varphi_x^2+\beta^2\varphi_{xx}^2+2\alpha\beta\varphi_{xx}\varphi_x+\alpha\varphi_{xx}+\beta\varphi_{xxx}\right)=0 \quad (8)$$

which is reduced to the equation

$$a\psi_t+d\psi_{xx}=0 \quad (9)$$

by the replacement

$$\psi=exp(\alpha\varphi+\beta\varphi_x) \quad (10)$$

The equation of type (5) is obtained from eq.(8) provided either

a) $d=c=\gamma=0$ (11)

or

b) $b=c=\beta=0$ (12)

In the first case, we obtain the equation

$$a(\alpha\varphi_t+\beta\varphi_{xt})+\left(\alpha^2\varphi_t\varphi_x+\alpha\beta(\varphi_{xt}\varphi_x+\varphi_t\varphi_{xx})+\beta^2\varphi_{xt}\varphi_{xx}+\alpha\varphi_{xt}+\beta\varphi_{xxt}\right)=0 \quad (13)$$

which is reduced to the equation

$$a\psi_t+b\psi_{xt}=0 \quad (14)$$

by the substitution

$$\psi=exp(\alpha\varphi+\beta\varphi_x) \quad (15)$$

In the second case, we have the equation

$$a(\alpha\varphi_t+\gamma\varphi_{tt})+d\left[(\alpha\varphi_x+\gamma\varphi_{tx})^2+\alpha\varphi_{xx}+\gamma\varphi_{t\,xx}\right] \quad (16)$$

which is reduced to the equation

$$a\psi_t+d\psi_{xx}=0 \quad (17)$$

by the replacement

$$\psi=exp(\alpha\varphi+\gamma\varphi_t) \quad (18)$$

The equation of type (6) is obtain from eq.(3) provided either

a) $c=d,\ \gamma=b=0$ (19)

or

b) $c=0,\ b=d\ \gamma+\beta=0$ (20)

In the first case, we obtain the equation

$$a(\alpha\varphi_t+\beta\varphi_{xt})+c\left[(\alpha\varphi_t+\beta\varphi_{xt})^2+d(\varphi_{tt}+\varphi_{xx})+(\alpha\varphi_x+\beta\varphi_{xx})^2+\beta(\varphi_{xxx}+\varphi_{xtt})\right]=0 \quad (21)$$

which is reduced to the equation

$$a\psi_t+c(\psi_{xx}+\psi_{tt})=0 \quad (22)$$

by the replacement

$$\psi=exp(\alpha\varphi+\beta\varphi_x) \quad (23)$$

In the second case we obtain the equation

$$a[\varphi_t+\beta(\varphi_{xt}-\varphi_{tt})]+b\left[\alpha^2(\varphi_t\varphi_x+\varphi_x^2)+\alpha\beta(\varphi_{xt}\varphi_x-\varphi_{tt}\varphi_x+\varphi_t\varphi_{xx}-\varphi_t\varphi_{xt}+2\varphi_{xx}\varphi_x-2\varphi_{tx}\varphi_x)+\right.$$
$$\left.+\beta^2(\varphi_{xt}\varphi_{xx}-\varphi_{tt}\varphi_{xx}+\varphi_{tt}\varphi_{tx}-\varphi_{xt}\varphi_{xx}^2-\varphi_{tx}^2-2\varphi_{xx}\varphi_{tx})+\alpha(\varphi_{xt}+\varphi_{xx})+\beta(\varphi_{xxx}-\varphi_{xtt})=0\right. \quad (24)$$

which is reduced to the equation

$$a\psi_t+b(\psi_{xt}+\psi_{xx})=0 \quad (25)$$

by the replacement

$$\psi=exp(\alpha\varphi+\beta\varphi_x-\beta\varphi_t) \quad (26)$$

The viscosity method

Let us investigate in detail equation (8). It is convenient to introduce a new notation

$$a\alpha=\alpha \qquad d\beta=k$$
$$a\beta=N \qquad d\beta=c \quad (27)$$

The equation (8) assumes the following form

$$L\varphi_t+N\varphi_{xt}+\frac{L^2k}{N^2}\varphi_x^2+k\varphi_{xx}^2+2\frac{Lk}{N}\varphi_{xx}\varphi_x+\frac{LC}{N}\varphi_{xx}+C\varphi_{xxx}=0 \quad (28)$$

Apparently, using the substitution $x'=\alpha x$ one can always equate the coefficients L and N. Then the equation (8) assumes the form

$$C(\varphi_{xxx}+\varphi_{xx})+2k\varphi_{xx}\varphi_x+k\varphi_{xx}^2+k\varphi_x^2+N(\varphi_t+\varphi_{xt})=0 \quad (29)$$

Accordingly, equations (9) and (10) rewritten as follows:

$N\varphi_t+C\psi_{xx}=0$ (9a)

$$\psi=exp\left[\frac{k}{c}(\varphi+\varphi_x)\right] \quad (10a)$$

Now let us note that the equation (29) can be used to find the solution of following equation:

$$2K\varphi_{xx}\varphi_x + k\varphi_{xx}^2 + k\varphi_x^2 + N(\varphi_t + \varphi_{xt}) = 0 \qquad (30)$$

Indeed, if φ_c is *a* solution of equation (29) then the function

$$\varphi = \lim_{C\to 0} \varphi_C \qquad (31)$$

can be considered as a formal solution of eq. (30).

Such a method of solving equations is often called "the viscosity method".

Let us write the explicit forms of some other equations, which can be solved by this method.

Making analogical (27) substitutions and equating the coefficients L_i and N_i (i=1,2,3,4)

one can rewrite the equations (13), (16), (21), (24) correspondingly and obtain ($c_i\to 0$) equations, which can be solved by the viscosity method

$$K_1(\varphi_{xt}\varphi_{xx} + \varphi_{xt}\varphi_x + \varphi_t\varphi_x) + N_1(\varphi_t + \varphi_{xt}) = 0 \qquad (13')$$

$$K_2(2\varphi_{tx}\varphi_x + \varphi_{tx}^2 + \varphi_x^2) + N_2(\varphi_t + \varphi_{xt}) = 0 \qquad (16')$$

$$K_3(2\varphi_{xx}\varphi_x + \varphi_{xx}^2 + \varphi_x^2 + 2\varphi_{xt}\varphi_t + \varphi_{xt}^2 + \varphi_t^2) + N_3(\varphi_t + \varphi_{xt}) = 0 \qquad (21')$$

$$K_4(\varphi_t\varphi_x + 2\varphi_x^2 - \varphi_{tt}\varphi_x + \varphi_t\varphi_{xx} - \varphi_t\varphi_{xt} + 2\varphi_{xx}\varphi_x - \varphi_{tx}\varphi_x - \varphi_{tt}\varphi_{xx} + \varphi_{tt}\varphi_{tx} - \varphi_{xt} + \varphi_{tx}^2 - \varphi_{xx}\varphi_{tx}) + \\ + N_4(\varphi_t + \varphi_{xt} - \varphi_{tt}) = 0 \qquad (24')$$

In conclusion we note, that general properties, exact solutions and soliton- like solutions of obtained equations will be discussed in another paper.

The author expresses his acknowledgements to Prof. V.I.Man'ko and Doc. V.V.Dodonov for useful discussions.

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KVANT DİNAMİK SİSTEMLƏRİ TƏSVİRİNDƏ QATILAŞMA METODU

Polinomşəkilli fərdi törəmələrdə qeyri-xətti differensial tənliklərin üç növünə baxılıb və araşdırılan tənlikləri xətti tənliklərə gətirən asılı dəyişənlərin əvəzolunmasının dəqiq ifadəsi alınmışdır. Həmçinin qatılaşma metodu ilə həll edilə bilən II dərəcəli fərdi törəməli bəzi differensial tənliklər alınmışdır.

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МЕТОД ВЯЗКОСТИ В ОПИСАНИИ КВАНТОВЫХ ДИНАМИЧЕСКИХ СИСТЕМ

Рассмотрены три типа нелинейных дифференциальных уравнений в частных производных полиномиального вида и получены явные выражения замен зависимых переменных, которые сводят исследуемые уравнения к линейным. Также получены некоторые дифференциальные уравнения в частных производных второго порядка, которые могут быть решены методом вязкости.

Received: 24.03.05