# LIMIT RELATION BETWEEN PSEUDO JACOBI POLYNOMIALS AND HERMIT POLYNOMIALS WITH A SHIFTED ARGUMENT

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In this paper, we prove a new limit relation between the pseudo-Jacobi polynomials and Hermite polynomials with shifted argument.

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## 1. NEW GENERALIZED FREE HAMILTONIAN

In paper [1], we proposed a new generalized free Hamiltonian with position-dependent mass $M = M(x)$ for the describing the dynamical quantum systems. This Hamiltonian has the form

$$H_0 = \frac{1}{4N} \sum_{i=1}^{N} \left( M^{\alpha} \hat{p}\, M^{\beta} \hat{p} M^{\gamma} + M^{\gamma} \hat{p}\, M^{\beta} \hat{p} M^{\alpha} \right). \tag{1}$$

It is compatible with Galilean invariance [2] and can be represented in the form

$$H_0 = -\frac{\hbar^2}{2M} \partial_x^2 + \frac{\hbar^2 M'}{2M^2} \partial_x + A_N \frac{{M'}^2}{M^3} + B_N \frac{M''}{M^2}, \tag{2}$$

where for the coefficients $A_N$ and $B_N$ we have the expressions

$$A_N = \frac{\hbar^2}{2N} A, \quad A = \sum_{i=1}^{N} (\alpha_i + \gamma_i + \alpha_i \gamma_i),$$
$$B_N = -\frac{\hbar^2}{4N} B, \quad B = \sum_{i=1}^{N} (\alpha_i + \gamma_i). \tag{3}$$

Note that all Hamiltonians used in the literature to describe the quantum dynamics of particles with mass dependent on the position [2-17]. Further, in the paper [1] on the basis of the Schrödinger equation, we constructed an exactly solvable model of a linear harmonic oscillator. The wave functions of this model are expressed in terms of pseudo Jacobi polynomials $P_n(x; \nu, N)$. The model mass function contains some parameter $a$. Purpose of this paper is to prove that in the limit $a \to \infty$ the pseudo Jacobi polynomials go over to the Hermite polynomials with a shifted argument $H_n(z - z_0)$.

## 2. BASİC FORMULAS

Pseudo Jacobi polynomials are defined in terms of hypergeometric functions as follows [18,19]

$$P_n(x; \nu, N) = \frac{(-2i)^n (-N + i\nu)_n}{(n - 2N - 1)_n} \, {}_2F_1 \left( \begin{matrix} -n, & n - 2N - 1 \\ & -N + i\nu \end{matrix} ; \frac{1 - ix}{2} \right), n = 0,1,2, \dots, N \tag{4}$$

and satisfy the orthogonality relation

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} (1 + x^2)^{-N-1} e^{2\nu \text{ arctanx}} P_m(x; \nu, N)\, P_n(x; \nu, N) =$$
$$= \frac{\Gamma(2N + 1 - 2n)\Gamma(2N + 2 - 2n)\, 2^{2n - 2N - 1} n!}{\Gamma(2N + 2 - n) |\Gamma(N + 1 - n + i\nu)|^2} \delta_{nm}. \tag{5}$$

We also write down for them a differential equation

$$(1 + x^2) y'' \big( x_1 + 2(\nu - Nx) y'(x) - n(n - 2N - 1) \big) y(x) = 0, \; y(x) = P_n(x; \nu, N) \tag{6}$$

Similar formulas for the Hermite polynomials are

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$$H_n(x) = (2x)^n {}_2F_0\left(-\frac{n}{2}, \quad -\frac{(n-1)}{2}; -\frac{1}{x^2}\right), \tag{7}$$

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx = 2^n (n!) \sqrt{\pi} \delta_{mn}, \tag{8}$$

$$y''(x) - 2xy'(x) + 2ny(x) = 0, \quad y(x) = H_n(x). \tag{9}$$

## 3. THEOREM

The following limit relation holds between the pseudo Jacobi and Hermite polynomials

$$\lim_{N\to\infty} 2^n N^{\frac{n}{2}} P_n\left(\frac{x}{\sqrt{N}};\ \nu\sqrt{N}, N\right) = H_n(x-\nu). \tag{10}$$

We will prove this theorem in two ways.
**Proof 1.** To prove (2), we will use the recurrence relations for the pseudo Jacobi and Hermite polynomials [18,19], which have the form

$$P_{n+1}(x;\nu,N) = A_n P_n(x;\nu,N) + B_n P_{n-1}(x;\nu,N), \tag{11}$$

$$H_{n+1}(z) = 2zH_n(z) - 2nH_{n-1}(z), \tag{12}$$

where

$$A_n(x,\nu) = x - \frac{\nu(N+1)}{(n-N-1)(n-N)},\ B_n(\nu) = -\frac{n(n-2N-2)(n-N-1-i\nu)(n-N+1+i\nu)}{(2n-2N-3)(n-N-1)^2(2n-2N-1)}. \tag{13}$$

Let be

$$Q_n = 2^n N^{n/2} P_n\left(\frac{x}{\sqrt{N}}; \nu\sqrt{N}, N\right) \text{ and } \quad \bar{Q}_n = \lim_{N\to\infty} Q_n. \tag{14}$$

Then from (11) we obtain the following recurrence relation for the polynomials $Q_n$

$$Q_{n+1} = \bar{A}_n Q_n + \bar{B}_n Q_{n-1}, \tag{15}$$

where

$$\bar{A}_n = 2\sqrt{N} A_n\left(\frac{x}{\sqrt{N}}; \nu\sqrt{N}\right), \bar{B}_n = 4NB_n(\nu\sqrt{N}). \tag{16}$$

Since $\lim_{N\to\infty} \bar{A}_n = 2(x-\nu)$ and $\lim_{N\to\infty} \bar{B}_n = -2n$, then passing to the limit $N \to \infty$ in (15) we come to

$$\bar{Q}_{n+1} = 2(x-\nu)\bar{Q}_n - 2n\bar{Q}_{n-1}. \tag{17}$$

And this coincides with the recurrence relation for the Hermite polynomials (12) for $z = x - \nu$. Hence, $\bar{Q}_n = H_n(x-\nu)$. This completes the proof of (10).

**Proof 2.** Let us now prove the relation (10) by the method of mathematical induction. To do this, we first write explicitly the pseudo Jacobi and Hermite polynomials for the first few values of $n$:

$$P_0(x;\nu,N) = 1,\ P_1(x;\nu,N) = x - \frac{\nu}{N},$$

$$P_2(x;\nu,N) = \left[x - \frac{\nu(N+1)}{N(N-1)}\right]\left(x - \frac{\nu}{N}\right) - \frac{N^2+\nu^2}{N^2(2N-1)},$$

$$H_0(z) = 1 \qquad H_1(z) = 2z \qquad H_2(z) = 4z^2 - 2. \tag{18}$$

Using the explicit form of polynomials (18), we can directly verify that for $n$ = 1 and 2, relation (10) is true, i.e.

$$\lim_{N\to\infty} 2\sqrt{N}\, P_1\left(\frac{x}{\sqrt{N}}; \nu\sqrt{N}, N\right) = H_1(x-\nu),$$

$$\lim_{N\to\infty} 2^2 N\, P_2\left(\frac{x}{\sqrt{N}}; \nu\sqrt{N}, N\right) = H_2(x-\nu). \qquad (19)$$

Assuming now that our relation (10) is true for n, we see that then it is true for $n + 1$ as well. Indeed, we have

$$\lim_{N\to\infty} 2^{n+1} N^{\frac{n+1}{2}} P_{n+1}\left(\frac{x}{\sqrt{N}}; \nu\sqrt{N}, N\right) =$$

$$= \lim_{N\to\infty} 2\sqrt{N} A_n\left(\frac{x}{\sqrt{N}}, \nu\sqrt{N}\right) \lim_{N\to\infty} 2^n N^{\frac{n}{2}} P_n\left(\frac{x}{\sqrt{N}}; \nu\sqrt{N}, N\right) +$$

$$+ \lim_{N\to\infty} 4N B_n\left(\nu\sqrt{N}\right) \lim_{N\to\infty} 2^{n-1} N^{\frac{n-1}{2}} P_{n-1}\left(\frac{x}{\sqrt{N}}; \nu\sqrt{N}, N\right) = 2zH_n(z) - 2nH_{n-1}(z).$$

Therefore, according to (18), we can write

$$\lim_{N\to\infty} 2^{n+1} N^{\frac{n+1}{2}} P_{n+1}\left(\frac{x}{\sqrt{N}}; \nu\sqrt{N}, N\right) = H_{n+1}(z), z = x - \nu.$$

Hence, the limit relation (10) is also true for all values of $n$. This completes the second proof of the theorem.

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