GENERALIZED HAMILTONIAN WITH POSITION-DEPENDENT MASS AND PSEUDO-JACOBI OSCILLATOR

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Exactly-solvable model of the quantum harmonic oscillator is proposed. In this work we propose a new generalized Hamiltonian, to describe variable mass systems. Wave functions of the stationary states and energy spectrum of the model are obtained through the solution of the corresponding Schrödinger equation with the positive position-dependent effective mass. We have shown that the wave functions of the stationary states of the model under consideration are expressed through the pseudo Jacobi polynomials $P_n(\xi;\ \nu,\ \bar{N})$. The parameter a of the model is quantized in terms of $\bar{N}$. As a consequence of it, the number of the its energy spectrum is finite. Under the limit $a \to \infty$ the system recovers the known non-relativistic quantum harmonic oscillator in the quantum mechanics. We also obtained the limiting relation between the pseudo Jacobi and Hermite polynomials.

Keywords: Position-dependent effective mass, new generalized free Hamiltonian, quantum harmonic oscillator, pseudo Jacobi polynomials, non-equidistant energy levels.
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1. INTRODUCTION

The number of exactly solvable problems in quantum mechanics is limited, but they play an important role in the study of the properties of various dynamical systems. This is, firstly, due to the fact that the exact solutions play the role of the foundation on which the solution of many other problems is built, secondly, they allow, from the point of view of symmetry, to understand their essence, and thirdly, they themselves can have directly various applications in many areas of theoretical physics. It should also be noted that exactly solvable problems are also interesting from the point of view of mathematics, since in many cases they can lead to the establishment of new properties of various special functions. For example, the problem of a harmonic oscillator, being exactly solvable, is widely used in atomic and nuclear physics, in the theory of crystals, in quantum field theory, etc. [1-4]. For this reason, the construction of exactly solvable quantum mechanical models, including models of a harmonic oscillator, for describing various physical systems has always attracted and continues to attract the attention of physicists [5-11].

On the other hand, various quantum mechanical systems described by the Schrödinger equation in cases where the Hamiltonian of the system contains the position-dependent mass [12-29]. These systems have found applications in a wide range of fields of the material science and condensed matter physics. A number of papers [14, 17, 19, 20, 22-29] are devoted to the construction of exactly solvable potentials for the Schrödinger equation with the position-dependent mass mass, and in [15] was obtained exact solution of the Dirac equation for a charged particle with position-dependent mass in the Coulomb field.

Our goal is to construct a quantum-mechanical exactly solvable model of a linear harmonic oscillator with the the position-dependent mass in an external uniform gravitational field. Our construction is based on the Schrödinger equation with a free Hamiltonian, generalizing the free Hamiltonian von Roos with the position-dependent mass. We show that the wave functions of our model are expressed in terms of pseudo-Jacobi polynomials. For this reason, we will call it the pseudo-Jacobi oscillator.

This article is organized as follows. Section 2 contains brief review of the nonrelativistic quantum-mechanical linear harmonic oscillator model.

2. NONRELATIVISTIC LINEAR HARMONIC OSCILLATOR WITH CONSTANT MASS

Let us write the one-dimensional Schrödinger equation describing the motion of a nonrelativistic quantum particle with constant mass m_0 in the external field $V(x)$.

It has the form

$$\left[\frac{\hat{p}^2}{2m_e} + V(x)\right]\psi(x) = E\psi(x), \qquad (2.1)$$

where $\hat{p} = -i\partial_x$ is the momentum operator. A linear harmonic oscillator with frequency ω corresponds to the following potential energy

$$V(x) = \frac{m_0\omega^2x^2}{2}. \qquad (2.2)$$

Let us rewrite equation (2.1) with potential (2.2) as

$$\frac{d^2\psi}{dx^2} + \frac{2m_0}{\hbar^2}\left(E - \frac{m_0\omega^2x^2}{2}\right)\psi = 0. \qquad (2.3)$$

The solution and energy spectrum of this equation are well known [1]

$$\psi_n^{0S}(x) = C_n^{0S}e^{-\frac{1}{2}\lambda_0^2x^2}H_n(\lambda_0 x),$$
$$E_n^{0S} = \hbar\omega\left(n + \frac{1}{2}\right),\ n = 0,1,2,\ldots, \qquad (2.4)$$

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where $H_n(x)$ are Hermite polynomials, and $\lambda_0 = \sqrt{m_0\omega/\hbar}$. Normalizing constants

$$C_n^{0S} = \frac{C_0^{0s}}{\sqrt{2^n n!}},$$

$$C_0^{0S} = \left(\frac{\lambda_0^2}{\pi}\right)^{1/4} = \left(\frac{m_0\omega}{\pi\hbar}\right)^{1/4} \quad (2.5)$$

found from the orthogonality condition for the Hermite polynomials [30, 31]

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x)H_n(x)dx = \sqrt{\pi}2^n n!\,\delta_{nm}. \quad (2.6)$$

3. GENERALIZED FREE HAMILTONIAN WITH THE POSITION-DEPENDENT MASS

In this paper, we will construct a new model of a nonrelativistic linear harmonic oscillator, namely with the position-dependent mass pseudo-Jacobi oscillator. It should be noted that the construction of models of quantum physical systems with the position-dependent mass $M \equiv M(x)$ starts with choosing the form of the free Hamiltonian to describe the position-dependent mass systems and with the subsequent selection of the mass function $M(x)$. The point is that due to the non-commutativity of the momentum operators $\hat{p} = -i\hbar\partial_x$ and the function $M(x)$, the question arises of their ordering in the expression for the free Hamiltonian

$$H_0 = \frac{1}{2M(x)}\hat{p}^2. \quad (3.1)$$

In this regard, we note that this issue was analyzed in [13], where it is proposed to restrict ourselves to a specific class of the form of the Hamiltonian with the position-dependent mass. According to this paper the free Hamiltonian operator must depend only on the mass function $M(x)$. Accordingly, it will take the following general form

$$H_0 = \frac{1}{2}\hat{p}\frac{1}{M(x)}\hat{p} + W_{\text{kin}}(x), \quad (3.2)$$

with the condition that the term W_{kin} be a functional of M, possibly involving its derivatives. Further the dimensional arguments require this term to be homogeneous of degree (-1) in M and of degree (-2) in x. Analyticity conditions precluding nonintegral powers of the derivatives of M and, finally, the condition that for a constant function $M(x) = m$ one recovers the usual expression, implying that the derivatives of M must appear with positive (integral) powers, lead to two possible terms only in W_{kin}:

$$W_{\text{kin}} = A_1\frac{M'^2}{M^3} + B_1\frac{M''}{M^2}. \quad (3.3)$$

In this paper is also stated that under the above conditions the most general free Hamiltonian is precisely the von Roos Hamiltonian of the form [12]

$$H_0 = \frac{1}{4}\left(M^\alpha\hat{p}\,M^\beta\hat{p}M^\gamma + M^\gamma\hat{p}\,M^\beta\hat{p}M^\alpha\right), \quad (3.4)$$

where the real parameters α, β and γ satisfy the natural condition $\alpha + \beta + \gamma = -1$, but otherwise they are arbitrary. The von Roos free Hamiltonian (3.4) has the form (3.2) with (3.3), where the coefficients A_1 and B_1 are

$$A_1 = \frac{1}{2}(\alpha + \gamma + \alpha\gamma), B_1 = -\frac{1}{4}(\alpha + \gamma). \quad (3.5)$$

In this section, we propose a new (generalized) free Hamiltonian to solve the problems with mass depending on the position as

$$H_0 = \frac{1}{4N}\sum_{i=1}^{N}\left(M^{\alpha_i}\hat{p}M^{\beta_i}\hat{p}M^{\gamma_i} + M^{\gamma_i}\hat{p}M^{\beta_i}\hat{p}M^{\alpha_i}\right), \quad (3.6)$$

where $N = 1,2,3\ldots$ is an arbitrary positive integer and the parameters $\alpha_i, \beta_i, \gamma_i$ $(i = 1,2,\ldots,N)$ satisfy the conditions(3.7)

$$\alpha_i + \beta_i + \gamma_i = -1, i = 1,2,\ldots,N.$$

The Hamiltonian (3.6) is more general than the Hamiltonian (3.4) von Roos [12] and the Hamiltonian

$$H_0 = \frac{1}{6}\left(\frac{1}{M}\hat{p}^2 + \hat{p}\frac{1}{M}\hat{p} + \hat{p}^2\frac{1}{M}\right), \quad (3.8)$$

proposed in [24]. Consequently, the free Hamiltonians used in the literature [12-29] for dynamical systems with the position-dependent mass are special cases (3.6). For example, for $N = 3$, $\alpha_1 = -1$, $\beta_1 = \gamma_1 = 0$, $\alpha_2 = \gamma_2 = 0$, $\beta_2 = -1$ and $\alpha_3 = -1$, $\beta_3 = \gamma_3 = 0$ from (3.6) follows (3.8). If we represent (3.6) in the form (3.2) and (3.3), i.e.

$$H_0 = -\frac{\hbar^2}{2M}\partial_x^2 + \frac{\hbar^2 M'}{2M^2}\partial_x + A_N\frac{M'^2}{M^3} + B_N\frac{M''}{M^2}, \quad (3.9)$$

then for the coefficients A_N and B_N we get the following expressions

$$A_N = \frac{\hbar^2}{2N}A, \quad A = \sum_{i=1}^{N}(\alpha_i + \gamma_i + \alpha_i\gamma_i),$$

$$B_N = -\frac{\hbar^2}{4N}B,\ \ B = \sum_{i=1}^{N}(\alpha_i + \gamma_i). \quad (3.10)$$

From the requirement that the Hamiltonian H_0 (3.6) (or (3.9)) be Hermitian, it follows that the coefficients A_N and B_N (3.10) must be real. Therefore, in the general case, the parameters $\alpha_i, \beta_i, \gamma_i$ $(i = 1,2,\dots,N)$ can be complex, provided that the relations $\gamma_i = \alpha_i^*$ $(i = 1,2,\dots,N)$ are satisfied, i.e. α_i and $\gamma_i$must be mutually complex conjugate.

Taking into account now (3.9), the Schrödinger equation for quantum systems with the position-dependent mass in the potential field $V(x)$ is written in the form

$$\left\{\partial_x^2 - \frac{M'}{M}\partial_x - \frac{A}{N}\frac{{M'}^2}{M^2} + \frac{B}{2N}\frac{M''}{M} + \frac{2M}{\hbar^2}[E - -V(x)]\right\}\psi(x) = 0. \quad (3.11)$$

4. PSEUDO-JACOBI OSCILLATOR WITH THE POSITION-DEPENDENT MASS

For building pseudo-Jacobi oscillator with the position-dependent mass we define position-dependent mass function $M(x)$ as follows

$$M(x) = \frac{a^2 m_0}{a^2+x^2}, \quad (4.1)$$

where $a > 0$ is some parameter with the dimension of length. Obviously, in the limit $a \to \infty$, the dependence of the mass (4.1) on the coordinate x disappears, i.e.

$$\lim_{a\to\infty} M(x) = m_0. \quad (4.2)$$

Let's write the potential energy of our model in the usual form

$$V(x) = \frac{M(x)\,\omega^2 x^2}{2}. \quad (4.3)$$

It is also clear that the following limit relations will take place both for the generalized free Hamiltonian (3.6) (or (3.9)) and for the potential energy of the model (4.3)

$$\lim_{a\to\infty} H_0 = -\frac{\hbar^2}{2m_0}\partial_x^2, \lim_{a\to\infty} V(x) = \frac{m_0\,\omega^2 x^2}{2}. \quad (4.4)$$

i.e., in the limit when the model parameter a tends to infinity ($a \to \infty$), these quantities coincide with the corresponding quantities of nonrelativistic quantum mechanics

Substituting the expression for mass (4.1) into (3.11), we obtain the Schrödinger equation describing the motion of our oscillator model

$$\left\{\partial_x^2 + \frac{2x}{a^2+x^2}\partial_x - \frac{A}{N}\frac{4x^2}{(a^2+x^2)^2} + \frac{B}{N}\frac{3x^2-a^2}{(a^2+x^2)^2} + \frac{2a^2 m_0}{\hbar^2(a^2+x^2)}\left[E - \frac{a^2 m_0 \omega^2 x^2}{2(a^2+x^2)}\right]\right\}\psi = 0. \quad (4.5)$$

In terms of the new dimensionless variable $\xi = \frac{x}{a}$, the equation (4.5) takes the form

$$\left(\partial_\xi^2 + \frac{\tilde{\tau}}{\sigma}\partial_\xi + \frac{\tilde{\sigma}}{\sigma^2}\right)\psi = 0, \quad (4.6)$$

where we have introduced the following notations $\tilde{\tau} = 2\xi$, $\sigma = 1 + \xi^2$, $\tilde{\sigma} = c_0 - c_2\xi^2$. For the coefficients c_0 and c_2 we have

$$c_0 = \frac{2a^2 m_0 E}{\hbar^2} - \frac{B}{N},$$
$$c_2 = \frac{a^4 {m_0}^2 \omega^2}{\hbar^2} - \frac{2m_0 a^2 E}{\hbar^2} - \frac{A_2}{N},$$

$$A_2 = \sum_{i=1}^{N}(\alpha_i + \gamma_i - 2\alpha_i\gamma_i). \quad (4.7)$$

Let us look for solution ψ of equation (4.6) as follows [32]:

$$\psi = \varphi(\xi)y(\xi),\ \varphi(\xi) = e^{\int\frac{\pi(\xi)}{\sigma(\xi)}d\xi}. \quad (4.8)$$

Here, $\pi(\xi)$ is an arbitrary polynomial of at most first degree and $\sigma \equiv \sigma(\xi)$. Then, by performing simple straightforward computations, one obtains the following second-order differential equation for $y \equiv y(\xi)$:

$$y'' + \frac{\bar{\tau}}{\sigma}y' + \frac{\bar{\sigma}}{\sigma^2}y = 0, \quad (4.9)$$

with

$$\bar{\tau} = \tilde{\tau} + 2\pi,\ \bar{\sigma} = \tilde{\sigma} + \pi^2 + \sigma\pi'.$$

It is seen that $\bar{\tau}(\xi)$ and $\bar{\sigma}(\xi)$ are polynomials, respectively, not higher than the first and second degrees. We now choose a polynomial $\pi(\xi)$ from the condition that the polynomial $\bar{\sigma}(\xi)$ be divided without remainder by $\sigma(\xi)$, i.e.

$$\bar{\sigma} = \lambda\sigma, \lambda = \text{const}. \quad (4.10)$$

Now, equation (4.9) takes the form

$$\sigma y'' + \bar{\tau}y' + \lambda y = 0. \quad (4.11)$$

Condition (4.10) gives a quadratic equation for the definition of a polynomial $\pi(\xi)$ and a constant λ:

$$\pi^2 - (\sigma' - \tilde{\tau})\pi - \mu\sigma + \tilde{\sigma} = 0,$$
$$\mu = \lambda - \pi'. \quad (4.12)$$

From here, we find

$$\pi = \frac{\sigma' - \tilde{\tau}}{2} + e\sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 + \mu\sigma - \tilde{\sigma}}\ ,\quad e = \pm 1. \quad (4.13)$$

In our case $\sigma' - \tilde{\tau} = 0$ and we have $\pi = e\sqrt{\mu\sigma - \tilde{\sigma}}$ or $\pi = e\sqrt{\mu - c_0 + (\mu + c_2)\xi^2}$. Since $\pi(\xi)$ is a polynomial, the discriminant of a polynomial of the second degree standing under the root (4.13) D must be equal to zero. The equation D = 0 allows us to find a constant μ. After determination μ, we find $\pi(\xi)$ by equation (4.12), then $\varphi(\xi)$, $\bar{\tau}(\xi)$ and λ with the help of (4.8), (4.9) and (4.12). In our case there are two possibility:

1) $\mu = c_0\ , \pi = e\, q_1\xi,\ q_1 = \sqrt{(c_0 + c_2)},$

2) $\mu = -c_2,\ \pi = eq_2,\ q_2 = \sqrt{-(c_0 + c_{2)}}\ ,$ (4.14)

Where $c_0 + c_2 = \frac{a^4 m_0{}^2\omega^2}{\hbar^2} - \frac{Q}{N}$, $Q = B + A_2 = 2\sum_{i=1}^{N}(\alpha_i + \gamma_i - \alpha_i\gamma_i)$. We will restrict ourselves to the case when $c_0 + c_2 > 0$. In this case, the physical meaning has the first expression for the polynomial $\pi(\xi)$. Moreover, for $\varphi(\xi)$ we obtain the following expression: $\varphi(\xi) = (1+\xi^2)^{\frac{eq_1}{2}}$. From the requirement of finiteness $\varphi(\xi)$ at points $\xi = \pm\infty$, i.e. from the condition $\lim_{\xi\to\pm\infty}\varphi(\xi) = 0$ (square integrability condition), we get $eq_1 < 0$. This means that $e = -1$ and, $\pi = -q_1\xi$. Thus, we get:

$$\varphi(\xi) = (1+\xi^2)^{-\frac{q_1}{2}},$$
$$q_1 = \sqrt{\frac{a^4 m_0{}^2\omega^2}{\hbar^2} - \frac{a}{N}} = \sqrt{\lambda_0^4 a^4 - \frac{a}{N}}. \qquad (4.15)$$

Now, taking into account that $\bar{\tau} = 2(1-q_1)\xi$ and $\lambda = \mu + \pi' = c_0 - q_1$, one can rewrite the equation (4.11) in the form

$$(1+\xi^2)y'' + 2(1-q_1)\xi y' + (c_0 - q_1)y = 0. \qquad (4.16)$$

Comparison (4.16) with the second order differential equation for the pseudo Jacobi polynomials $\bar{y} \equiv P_n(\xi;\ \nu,\ \bar{N})$:

$$(1+\xi^2)\bar{y}'' + 2(\nu - \bar{N}\xi)\bar{y}' + n(2\bar{N} - n + 1)\bar{y} = 0,\ n = 0,1,2,3 \ldots, \bar{N} \qquad (4.17)$$

gives us the relations

$$\nu = 0,\ q_1 = \bar{N} + 1,$$
$$c_0 - q_1 = n(2\bar{N} + 1 - n),\ \bar{N} = 0,1,2,3, \ldots \qquad (4.18)$$

These relations lead to quantization of arbitrary parameter a being of position dimensions and introduced in the framework of definition of the position-dependent mass $M(x)$ (4.1):

$$a \equiv a_N = \left[(\bar{N}+1)^2 + \frac{Q}{N}\right]^{1/4}. \qquad (4.19)$$

Therefore, position-dependent effective mass $M(x)$ is quantized as follows:

$$M(x) \equiv M_N(x) = \frac{\sqrt{(\bar{N}+1)^2 + \frac{Q}{N}}}{\sqrt{(\bar{N}+1)^2 + \frac{Q}{N} + \lambda_0^2 x^2}}\, m_0. \qquad (4.20)$$

Taking this hidden feature of position-dependent effective mass $M(x)$, one obtains the following exact expressions for the energy spectrum

$$E \equiv E_n = \frac{\hbar^2}{2m_0 a^2}[n(2\bar{N} + 1 - n) + \bar{N} + 1 + Q/N], \qquad (4.21a)$$

or

$$E_n = \frac{1}{2}\hbar\omega\frac{n(2\bar{N}+1-n) + \bar{N} + 1 + Q/N}{\sqrt{(\bar{N}+1)^2 + Q/N}},\quad n = 0,1,2,3 \ldots, N. \qquad (4.21b)$$

Thus, exact polynomial of equation (4.16) are expressed by the pseudo Jacobi polynomials, i. e.

$$y(\xi) \equiv y_n(\xi) \equiv P_n(\xi;\nu,\bar{N}). \qquad (4.22)$$

Now, taking into account (4.8), (4.15) and (4.22) one obtains the following expression for the wave functions of our model

$$\psi(\xi) \equiv \psi_{Nn}(\xi) = C_{Nn}(1+\xi^2)^{-\frac{q_1}{2}} P_n(\xi;0,\bar{N}). \qquad (4.23)$$

Let's express them through the variable x:

$$\psi_{Nn}(x) = C_{Nn}\left(1+\frac{\lambda_0^2x^2}{\sqrt{(\bar{N}+1)^2+Q/_N}}\right)^{-\frac{\bar{N}+1}{2}} P_n\left(\frac{\lambda_0 x}{\sqrt[4]{(\bar{N}+1)^2+Q/_N}};0,\bar{N}\right) \tag{4.24}$$

Normalizing constants $(n = 0,1,2,3\ldots,N)$

$$C_{Nn} = 2^{\bar{N}-n}\sqrt{\frac{\lambda_0}{\pi n!}}\left[(\bar{N}+1)^2+Q/_N\right]^{-\frac{1}{8}}\frac{\Gamma(\bar{N}+1-n)}{\Gamma(2\bar{N}+1-2n)}\sqrt{\frac{\Gamma(2\bar{N}+2-n)}{2\bar{N}+1-2n}} \tag{4.25}$$

we find from the condition that the wave functions (4.24) are orthonormal

$$\int_{-\infty}^{\infty}\psi_{Nn}^*(x)\psi_{Nm}(x)dx = \delta_{nm}. \tag{4.26}$$

In calculating the integral in (4.26), we used the orthogonality condition for the pseudo Jacobi polynomials [31]

$$\int_{-\infty}^{\infty}(1+x)^{-N-1}e^{2\nu\mathrm{arctg}x}P_n(x;\nu,N)P_m(x;\nu,N)dx = d_n^2\delta_{nm}, \tag{4.27}$$

where d_n^2- is the square of the norm of the pseudo Jacobi polynomials and is equal to

$$d_n^2 = \pi n!\,2^{2n-2N}\frac{\Gamma(2N+1-2n)\Gamma(2N+2-2n)}{\Gamma(2N+2-n)|\Gamma(N+1-n+i\nu)|^2}. \tag{4.28}$$

In conclusion of this section, we also present the form of the wave functions, explicitly indicating the dependences on the parameter a (4.19) of the model

$$\psi_{Nn}(x) = C_{Nn}\left(1+\frac{x^2}{a^2}\right)^{-\frac{1}{2}\sqrt{\lambda_0^4a^4-Q/_N}} P_n\left(\frac{x}{a};0,\sqrt{\lambda_0^4a^4-Q/_N}-1\right). \tag{4.29}$$

5. CONCLUSION

In this paper, we have constructed an exactly solvable linear harmonic oscillator model with the position-dependent mass. The main point of this article is the proposal of a new and the most general Hamiltonian for the quantum dynamical systems with the position-dependent mass. This Hamiltonian contains, in the form of special cases, the Hamiltonians used in the literature. One feature of the psevdo Jacobi oscillator is that the number of its energy levels is finite. This is due to the fact that the depth of the pseudo-parabolic oscillatory well $V_0 > 0$ is finite. This depth is

$$\lim_{x\to\pm\infty}V(x) = \lim_{x\to\pm\infty}\frac{M(x)\omega^2x^2}{2} = \frac{1}{2}\hbar\omega\sqrt{(\bar{N}+1)^2+Q/N} \equiv V_0. \tag{5.1}$$

The second feature is related to the fact that the energy levels are not equidistant. The minimum and maximum energy values are respectively

$$E_{N0} = \frac{\hbar\omega}{2}\frac{(\bar{N}+1)^2+Q/N}{\sqrt{(\bar{N}+1)^2+Q/N}} \quad\text{and}\quad E_{N\bar{N}} = \frac{\hbar\omega}{2}\frac{\bar{N}(\bar{N}+2)+1+Q/N}{\sqrt{(\bar{N}+1)^2+Q/N}}. \tag{5.2}$$

The wave functions of the constructed model of the oscillator are expressed in terms of pseudo Jacobi polynomials. In the limit $a\to\infty$, all quantities (equation of motion, wave functions, energy spectrum) of this model transform into the corresponding quantities of an ordinary linear harmonic oscillator with constant mass. For example, for energy levels (4.21) and wave functions (4.24) we have the following limit expressions

$$\lim_{a\to\infty}E_n = \hbar\omega(n+1/2), \tag{5.3}$$

$$\lim_{a\to\infty}\psi_{Nn}(x) = \sqrt{\frac{\lambda_0}{2^n n!\sqrt{\pi}}}H_n(\lambda_0 x)e^{-\frac{1}{2}\lambda_0^2x^2}. \tag{5.4}$$

It is clear from formula (5.4) that there is a limit relation between the pseudo Jacobi and Hermite polynomials with a shifted argument. We will prove in the next paper that it has the form

$$\lim_{N\to\infty}N^{\frac{n}{2}}P_n\left(\frac{x}{\sqrt{N}};\,\nu\sqrt{N},N\right) = \frac{1}{2^n}H_n(x-\nu).$$

(5.5)

Let us make one more remark about the properties of the pseudo Jacobi oscillator, connected with the form of the generalized free Hamiltonian (3.6) (or (3.9)). In obtaining solutions (4.21) and (4.24), we assumed that the following inequality holds:

$$\frac{a^4 m_0{}^2 \omega^2}{\hbar^2} - \frac{Q}{N} > 0, \qquad (5.6)$$

where $Q = 2\sum_{i=1}^{N}(\alpha_i + \gamma_i - \alpha_i\gamma_i)$. However, for certain values of the parameters $\alpha_{i\,,}\gamma_i$ $(i = 0,1,2,3\ldots,N$ this inequality may not hold, i.e., the pseudo Jacobi oscillator for certain energy values may not have a discrete spectrum ...

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