ON THE EXACT SOLUTION OF THE CONFINED POSITION-DEPENDENT MASS HARMONIC OSCILLATOR MODEL WITH THE KINETIC ENERGY OPERATOR COMPATIBLE WITH GALILEAN INVARIANCE UNDER THE HOMOGENEOUS GRAVITATIONAL FIELD

A.M. MAMMADOVA
Institute of Physics, Azerbaijan National Academy of Sciences
H. Javid av. 131, AZ1143, Baku, Azerbaijan
E-mail: a.mammadova@physics.science.az

Exactly-solvable confined model of the quantum harmonic oscillator under the external gravitational field is studied. Confinement effect is achieved thanks to the effective mass changing with position. Nikiforov-Uvarov method is applied for solving exactly corresponding Schrödinger equation. Analytical expressions of the wavefunctions of the stationary states and energy spectrum are obtained.

Keywords: Harmonic oscillator, gravitational field, position-dependent mass.
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1. INTRODUCTION

Quantum systems with position-dependent effective mass have been the subject of many attracting studies in recent years [1-4]. The Schrödinger equation corresponding to such systems with non-constant mass provides interesting and useful solutions for the description of them. At the same time, behavior of the quantum system influenced from attached external field is also within the attraction of the scientists working on this and related research topics [5,6]. Main reason is that external field attached to the quantum system under consideration can thoroughly change its main properties. Then, such an effect also can open for studying many of previously hidden aspects regarding them.

In present paper, we study an oscillator model that is under the influence of the external homogeneous gravitational field. Already, same model exhibiting confinement effect was studied in detail and results have been published in [7]. Then, taking into account importance of the appearance of the external field, we decided to obtain more general solutions by extending free motion in the confined oscillator potential to the similar motion, but through taking into account external homogeneous gravitational field. We are able to obtain analytical expressions of the wavefunctions corresponding to our model under study as well as its energy spectrum. In our studies, we preserved general definition of the kinetic operator, which is still compatible with the Galilean invariance. Correctness of the obtained analytical expressions is proven via their correct reduce to the known non-relativistic results under the certain limit relations.

We structured our paper as follows: in Section 2, basic known information is provided for the non-relativistic quantum harmonic oscillator. Briefly, analytical solutions of the Schrödinger equation for it are presented within the canonical approach. Analytical solutions cover both wave functions of the stationary states of the quantum oscillator itself as well as same model but under the external homogeneous gravitational field. It is shown that both wave functions are expressed via the Hermite polynomials, but wave function for the model suppressed to the external homogeneous gravitational field differs from free harmonic oscillator with the shifted position x. Analytical expressions of the energy spectrum of both models also have similar behavior – both of them are equidistant. However, energy spectrum of the model under the external homogeneous gravitational field differs with some shifted constant parameter that appears as a result of the applied external field. It is shown that analytical expression of the wave functions and energy spectrum of the model with an applied external homogeneous gravitational field easily recovers the model of the non-relativistic oscillator within the canonical approach for case of the disappearance of the external field. Section 3 is devoted to the confined position-dependent mass harmonic oscillator model under the homogeneous gravitational field. In order achieve the confinement effect, we replaced constant effective mass of the model under study with the effective mass that varies with position x. Then, aiming to preserve Hermiticity property of the Hamiltonian of the model, we also replaced its kinetic energy operator with the kinetic energy operator compatible with Galilean invariance. Both analytical expressions of the wave functions and energy spectrum of the confined position-dependent mass harmonic oscillator model with the kinetic energy operator compatible with Galilean invariance and same model but under the homogeneous gravitational field are presented here. Final section contains some brief discussions and possible limit relations between the models presented here.

2. NON-RELATIVISTIC QUANTUM HARMONIC OSCILLATOR WITHIN THE CANONICAL APPROACH – EXACT SOLUTIONS THE MODEL WITH BOTH CASES OF ABSENCE AND EXISTENCE

131, H.Javid ave, AZ-1143, Baku
ANAS, Institute of Physics
E-mail: jophphysics@gmail.com

OF THE EXTERNAL GRAVITATIONAL FIELD $V(x) = m_0 g x$

Quantum-mechanical solution for the one-dimentional harmonic oscillator with wavefunctions, which have to vanish at infinity can be obtained by solving exactly one-dimensional stationary Schrödinger equation in the position representation

$$\left[\frac{\hat{p}_x^2}{2m} + V(x)\right]\psi(x) = E\psi(x), \qquad (2.1)$$

with non-relativistic harmonic oscillator potential

$$V(x) = \frac{m_0\omega^2 x^2}{2}. \qquad (2.2)$$

Here, m_0 and ω are constant effective mass and angular frequency of the non-relativistic quantum harmonic oscillator, and one-dimentional momentum operator $\hat{p}_x$ is defined in canonical approach as

$$\hat{p}_x = -i\hbar\frac{d}{dx}. \qquad (2.3)$$

Taking into account (2.2) and (2.3) in (2.1) we have

$$\frac{d^2\psi}{dx^2} + \frac{2m_0}{\hbar^2}\left(E - \frac{m_0\omega^2 x^2}{2}\right)\psi = 0. \qquad (2.4)$$

Solving this equation exactly, we obtain the following expression for the energy spectrum:

$$E \equiv E_n = \hbar\omega\left(n + \frac{1}{2}\right), \quad n = 0,1,2,\ldots \qquad (2.5).$$

It is possible also to show that wavefunctions of the stationary states the model under consideration in the position representation obtained from (2.4) are

$$\psi \equiv \psi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m_0\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}} H_n\left(\sqrt{\frac{m_0\omega}{\hbar}}x\right), \qquad (2.6)$$

where $H_n(x)$ are Hermite polynomials defined in terms of hypergeometric function as follows

$$H_n(x) = (2x)^n {}_2F_0\left(-\frac{n}{2}, -\frac{n-1}{2}; -; \frac{1}{x^2}\right). \qquad (2.7)$$

Let's now consider the model of a linear harmonic oscillator (2.2) in external homogeneous gravitational field. Then, the potential of the harmonic oscillator is

$$V(x) = \frac{m_0\omega^2 x^2}{2} + m_0 g x. \qquad (2.8)$$

Now we need to solve the following Schrödinger equation:

$$\left[\frac{\hat{p}_x^2}{2m_0} + \frac{m_0\omega^2 x^2}{2} + m_0 g x\right]\psi(x) = E\psi(x). \qquad (2.9)$$

Note that all calculations are still in a canonical approach. Therefore, one-dimensional momentum operator can be written as (2.3). We have

$$\frac{d^2\psi}{dx^2} + \left[\frac{2m_0 E}{\hbar^2} - \frac{{m_0}^2\omega^2}{\hbar^2}(x + x_0)^2 + \frac{g^2}{\omega^4}\right]\psi = 0, \qquad (2.10)$$

where,

$$x_0 = \frac{g}{\omega^2}. \qquad (2.11)$$

We can rewrite (2.10) as

$$\frac{d^2\psi}{dx^2} + \left[\frac{2m_0\tilde{E}}{\hbar^2} - \frac{{m_0}^2\omega^2}{\hbar^2}(x + x_0)^2\right]\psi = 0, \qquad (2.12)$$

$$\tilde{E} = E + \frac{\hbar^2 g^2}{2m_0\omega^4}. \qquad (2.13)$$

Analytical solution of the equation (2.12) leads to explicit expression of the discrete equidistant energy spectrum:

$$E \equiv E_n^g = \hbar\omega\left(n + \frac{1}{2}\right) - \frac{\hbar^2 g^2}{2m_0\omega^4}, \; n = 0,1,2,\ldots. \qquad (2.14)$$

The corresponding wavefunctions are

$$\psi \equiv \psi_n^g(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m_0\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m_0\omega\left(x+\frac{g}{\omega^2}\right)^2}{2\hbar}} H_n\left(\sqrt{\frac{m_0\omega}{\hbar}}\left(x+\frac{g}{\omega^2}\right)\right). \tag{2.15}$$

One easily observes that

$$\psi_n^g(x) = \psi_n(x - x_0).$$

Under the case $g = 0$ both energy spectrum E_n^g (2.14) and wavefunctions $\psi_n^g(x)$ (2.15) correctly recover energy spectrum E_n (2.5) and wavefunctions $\psi_n(x)$ (2.6).

3. CONFINED POSITION-DEPENDENT MASS HARMONIC OSCILLATOR MODEL UNDER THE HOMOGENEOUS GRAVITATIONAL FIELD

Recently, we considered the quantum harmonic oscillator problem confined in the finite region, which effective mass varied with position $m_0 \to M(x)$ and its kinetic energy operator was compatible with Galilean invariance [8]:

$$\hat{H}_0^{GI} = -\frac{\hbar^2}{6}\left[\frac{1}{M(x)}\frac{d^2}{dx^2} + \frac{d}{dx}\frac{1}{M(x)}\frac{d}{dx} + \frac{d^2}{dx^2}\frac{1}{M(x)}\right]. \tag{3.1}$$

We introduced confined harmonic oscillator potential as

$$V(x) = \begin{cases} \frac{M(x)\omega^2 x^2}{2}, & |x| < a, \\ \infty, & |x| \geq a, \end{cases} \tag{3.2}$$

and then solved exactly the Schrödinger equation corrsponding to the following Galilean invariant Hamiltonian:

$$\hat{H}^{GI} = -\frac{\hbar^2}{2M}\left[\frac{d^2}{dx^2} - \frac{M'}{M}\frac{d}{dx} - \frac{1}{3}\frac{M''}{M} + \frac{2}{3}\left(\frac{M'}{M}\right)^2\right] + \frac{M(x)\omega^2 x^2}{2}. \tag{3.3}$$

Here, we also defined position-dependent effective mass $M(x)$ via the following analytical expression:

$$M \equiv M(x) = \frac{a^2 m_0}{a^2 - x^2}. \tag{3.4}$$

We obtained that energy spectrum E_n^{GI} is non-equidistant and has the following expression:

$$E_n^{GI} = \hbar\omega\left(n + \frac{1}{2}\right) + \frac{\hbar^2}{2m_0 a^2}n(n+1) + \frac{\hbar^2}{3m_0 a^2}, \tag{3.5}$$

whereas the wavefunctions of the stationary states ψ^{GI} are expressed through the Gegenbauer polynomials by the following manner:

$$\tilde{\psi}^{GI}(x) = c_n^{GI}\left(1 - \frac{x^2}{a^2}\right)^{\frac{m_0\omega a^2}{2\hbar}} C_n^{\left(\frac{m_0\omega a^2}{\hbar} + \frac{1}{2}\right)}\left(\frac{x}{a}\right). \tag{3.6}$$

Here, Gegenbauer polynomials $C_n^{\bar{\lambda}}(x)$ are defined in terms of the ${}_2F_1$ hypergeometric functions as follows:

$$C_n^{(\bar{\lambda})}(x) = \frac{(2\bar{\lambda})_n}{n!}\ {}_2F_1\left(\begin{matrix}-n, n+2\bar{\lambda} \\ \bar{\lambda}+\frac{1}{2}\end{matrix};\ \frac{1-x}{2}\right),\ \bar{\lambda} \neq 0.$$

*N*ormalization factor c_n^{GI} is obtained from the orthogonality relation for the Gegenbauer polynomials and its exact expression is the following:

$$c_n^{GI} = 2^{\frac{m_0\omega a^2}{\hbar}}\Gamma\left(\frac{m_0\omega a^2}{\hbar} + \frac{1}{2}\right)\sqrt{\frac{\left(n + \frac{m_0\omega a^2}{\hbar} + \frac{1}{2}\right)n!}{\pi a \Gamma\left(n + \frac{2m_0\omega a^2}{\hbar} + 1\right)}}.$$

Now we can explore confined position-dependent mass harmonic oscillator model under the homogeneous gravitational field. First of all, we introduce external field to confined harmonic oscillator potential (3.2) as follows:

$$V(x)=\begin{cases}\frac{M(x)\omega^2x^2}{2}+M(x)gx, & |x|\le a,\\ \infty, & |x|>a.\end{cases} \tag{3.7}$$

Taking into account analytical definition of the position-dependent effective mass $M(x)$ (3.4) we need to solve the following Schrödinger equation:

$$\left[\frac{d^2}{dx^2}-\frac{2x}{a^2-x^2}\frac{d}{dx}+\frac{2}{3}\frac{4x^2}{(a^2-x^2)^2}-\frac{1}{3}\frac{2}{a^2-x^2}-\frac{1}{3}\frac{8x^2}{(a^2-x^2)^2}\right]\psi+\frac{2M}{\hbar^2}\left[E-\frac{M\omega^2x^2}{2}-Mgx\right]\psi=0 \tag{3.8}$$

Introduction of the new dimensionless variable ξ as:

$$\xi=\frac{x}{a},\ \ \frac{d}{dx}=\frac{1}{a}\frac{d}{d\xi},\ \ \frac{d^2}{dx^2}=\frac{1}{a^2}\frac{d^2}{d\xi^2}$$

and

$$c_0=\frac{2m_0a^2E}{\hbar^2},\ \ c_1=\frac{2{m_0}^2ga^3}{\hbar^2},\quad c_2=c_0+{\lambda_0}^4a^4,$$

leads to:

$$\psi''-\frac{2\xi}{1-\xi^2}\psi'+\left(\frac{c_0}{1-\xi^2}-\frac{(c_2-c_0)\xi^2}{(1-\xi^2)^2}-\frac{2}{3}\frac{1}{1-\xi^2}-\frac{c_1\xi}{(1-\xi^2)^2}\right)\psi=0.$$

Taking into account

$$\frac{c_0}{1-\xi^2}-\frac{(c_2-c_0)\xi^2}{(1-\xi^2)^2}-\frac{2}{3}\frac{1}{1-\xi^2}-\frac{c_1\xi}{(1-\xi^2)^2}=\frac{c_0-\frac{2}{3}-c_1\xi-\left(c_2-\frac{2}{3}\right)\xi^2}{(1-\xi^2)^2},$$

We get

$$\psi''-\frac{2\xi}{1-\xi^2}\psi'+\frac{c_0-\frac{2}{3}-c_1\xi-\left(c_2-\frac{2}{3}\right)\xi^2}{1-\xi^2}\psi=0. \tag{3.9}$$

To solve this equation exactly we can apply Nikiforov-Uvarov method [9], which can be applied to the following second order differential equations:

$$\psi''+\frac{\tilde\tau}{\sigma}\psi'+\frac{\tilde\sigma}{\sigma^2}\psi=0. \tag{3.10}$$

Here, it is assumed that σ and $\tilde\sigma$ are arbitrary polynomials of at most second degree and $\tilde\tau$ is an arbitrary polynomial of at most first degree. The following comparison allows to say that Nikiforov-Uvarov method is applicable to exact solution of eq.(3.9):

$$\tilde\tau=-2\xi,\ \ \sigma=1-\xi^2,\quad \tilde\sigma=c_0-\frac{2}{3}-c_1\xi-\left(c_2-\frac{2}{3}\right)\xi^2 \tag{3.11}$$

We look for expression of ψ as:

$$\psi=\varphi(\xi)y,\qquad \text{where}\qquad \varphi=e^{\int\frac{\pi(\xi)}{\sigma(\xi)}d\xi}. \tag{3.12}$$

Via simple computations one finds that

$$\psi'=\frac{\pi}{\sigma}\varphi y+\varphi y',$$

$$\psi''=\frac{\pi'\sigma-\pi\sigma+\pi^2}{\sigma^2}\varphi y+\frac{2\pi}{\sigma}\varphi y'+\varphi y''.$$

Taking these computations into account in (3.9) leads to the equation for $y(\xi)$:

$$y''+\frac{2\pi+\tilde\tau}{\sigma}y'+\frac{\tilde\sigma+\pi^2+\pi(\tilde\tau-\sigma')+\pi'\sigma}{\sigma^2}y=0, \tag{3.13}$$

where

$$\tau=2\pi+\tilde\tau,\qquad \bar\sigma=\tilde\sigma+\pi^2+\pi(\tilde\tau-\sigma')+\pi'\sigma. \tag{3.14}$$

We can rewrite (3.13), as

$$y'' + \frac{\tau}{\sigma}y' + \frac{\bar{\sigma}}{\sigma^2}y = 0 \tag{3.15}$$

Assuming that

$$\bar{\sigma} = \lambda\sigma, \qquad \lambda = \text{const}, \qquad \mu = \lambda - \pi', \tag{3.16}$$

we have

$$\lambda\sigma = \tilde{\sigma} + \pi^2 + \pi(\tilde{\tau} - \sigma') + \pi'\sigma,$$

which requires to solve the following quadratic equation:

$$\pi^2 + (\tilde{\tau} - \sigma')\pi + \tilde{\sigma} - \mu\sigma = 0.$$

Taking into account:

$$\sigma' = -2\xi\,, \qquad \tilde{\tau} - \sigma' = 0,$$

we find that

$$\pi = \varepsilon_1\sqrt{\mu\sigma - \tilde{\sigma}} = \varepsilon_1\sqrt{\mu + \frac{2}{3} - c_0 + c_1\xi + \left(c_2 - \left(\mu + \frac{2}{3}\right)\right)\xi^2}, \qquad \varepsilon_1 = \pm 1. \tag{3.17}$$

After some computations:

$$\mu = \frac{c_2 + c_0 + \varepsilon_2\sqrt{(c_2 - c_0)^2 - c_1^2}}{2} - \frac{2}{3}, \;\; \varepsilon_2 = \pm 1, \tag{3.18}$$

$$c_2 - \left(\mu + \frac{2}{3}\right) = \frac{c_2 - c_0 - \varepsilon_2\sqrt{(c_2 - c_0)^2 - c_1^2}}{2} = \kappa. \tag{3.19}$$

Substituting (3.18) & (3.19) at (3.17), we obtain the following expressions for π, τ and λ:

$$\pi = \varepsilon_1\left(\sqrt{\kappa}\xi + \frac{c_1}{2\sqrt{\kappa}}\right), \tag{3.20}$$

$$\tau = 2\varepsilon_1\left(\sqrt{\kappa}\xi + \frac{c_1}{2\sqrt{\kappa}}\right) - 2\xi = 2\left(\varepsilon_1\sqrt{\kappa} - 1\right)\xi + \varepsilon_1\frac{c_1}{\sqrt{\kappa}}. \tag{3.21}$$

$$\lambda = \mu + \pi' = \frac{c_2 + c_0 + \varepsilon_2\sqrt{(c_2 - c_0)^2 - c_1^2}}{2} - \frac{2}{3} + \varepsilon_1\sqrt{\kappa}. \tag{3.22}$$

Taking into account (3.20) at (3.12) we have to compute the following integral

$$\varphi(\xi) = e^{\int\frac{\pi(\xi)}{\sigma(\xi)}d\xi} = e^{\varepsilon_1\sqrt{\kappa}\int\frac{\xi}{1-\xi^2}d\xi}\,e^{\varepsilon_1\frac{c_1}{2\sqrt{\kappa}}\int\frac{1}{1-\xi^2}d\xi},$$

that gives for us

$$\varphi(\xi) = (1 - \xi)^{-\kappa_1}(1 + \xi)^{-\kappa_2},$$

$$\kappa_{1,2} = \frac{1}{2}\varepsilon_1\left(\sqrt{\kappa} \pm \frac{c_1}{2\sqrt{\kappa}}\right).$$

Finiteness of the $\varphi(\xi)$ at singular points $\xi = \pm 1$, i. e. the condition $\lim_{\xi\to\pm 1}\varphi(\xi) = const$ leads to

$$\varepsilon_1 = \varepsilon_2 = -1, \qquad \kappa > 0, \;\; \kappa_{1,2} \leq 0.$$

Therefore, we have

$$\mu = \frac{c_2 + c_0 - \sqrt{(c_2 - c_0)^2 - c_1^2}}{2} - \frac{2}{3}, \tag{3.23}$$

$$\pi = -\sqrt{\kappa}\xi - \frac{c_1}{2\sqrt{\kappa}}, \tag{3.24}$$

$$\kappa = \frac{c_2 - c_0 + \sqrt{(c_2 - c_0)^2 - c_1^2}}{2}, \tag{3.25}$$

$$\sqrt{\kappa} = -\kappa_1 - \kappa_2, \tag{3.26}$$

$$\kappa_{1,2} = -\frac{1}{2}\left(\sqrt{\kappa} \pm \frac{c_1}{2\sqrt{\kappa}}\right). \tag{3.27}$$

Then, expression of the wavefunction ψ also will have the following exact expression:

$$\psi = \varphi(\xi)y = (1 - \xi)^{-\kappa_1}(1 + \xi)^{-\kappa_2}y. \tag{3.28}$$

We also obtain the following expressions for $\pi(\xi)$, τ and λ in terms of κ_1 and κ_2:

$$\pi = (\kappa_1+\kappa_2)\xi + \frac{c_1}{2(\kappa_1+\kappa_2)}, \tag{3.29}$$

$$\tau = 2(\kappa_1+\kappa_2-1)\xi + \frac{c_1}{\kappa_1+\kappa_2}, \tag{3.30}$$

$$\lambda = \frac{c_2+c_0-\sqrt{(c_2-c_0)^2-c_1^2}}{2} - \frac{2}{3} + \kappa_1+\kappa_2 . \tag{3.31}$$

Then, eq. (3.15) will have the form as follows:

$$\sigma y'' + \tau y' + \lambda y = 0. \tag{3.32}$$

Function $y(\xi)$ should be finite at values $\xi = \pm 1$. Therefore, we have to find its polynomial solutions. For this reason, we compare the following exact expression of eq. (3.32)

$$(1-\xi^2)y'' + \left[2(\kappa_1+\kappa_2-1)\xi + \frac{c_1}{\kappa_1+\kappa_2}\right]y' + \lambda y = 0$$

with the following second order differential equation for the Jacobi polynomials

$$(1-x^2)\bar{y}'' + [\beta-\alpha-(\alpha+\beta+2)x]\bar{y}' + n(n+\alpha+\beta+1)\bar{y} = 0,$$

where, $\bar{y} = P_n^{(\alpha,\beta)}(x)$ are Jacobi polynomials defined in terms of the ${}_2F_1$ hypergeometric functions as follows:

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n\, {}_2F_1}{n!}\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix};\ \frac{1-x}{2}\right), \quad \alpha;\beta \neq -1/2,$$

$$\alpha+\beta = -2(\kappa_1+\kappa_2),$$

$$\alpha+\beta = \frac{c_1}{\kappa_1+\kappa_2}.$$

Some computations lead us to the following non-equidistant energy spectrum:

$$E \equiv E_n^{gGI} = \hbar\omega\sqrt{\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{g^2}{a^2\omega^4}}}\left(n+\frac{1}{2}\right) + \frac{\hbar^2}{2m_0a^2}n(n+1) + \frac{\hbar^2}{3m_0a^2} -$$

$$- m_0\omega^2a^2\left(\frac{1}{2}-\sqrt{\frac{1}{4}-\frac{g^2}{a^2\omega^4}}\right). \tag{3.33}$$

Wavefunctions of the stationary states have form:

$$\psi \equiv \psi_n^{gGI}(x) = c_n^{gL}\left(1-\frac{x}{a}\right)^{-\kappa_1}\left(1+\frac{x}{a}\right)^{-\kappa_2} P_n^{(-2\kappa_1,-2\kappa_2)}\left(\frac{x}{a}\right). \tag{3.34}$$

The normalization factor c_n^{gGI}

$$c_n^{gGI} = \frac{1}{2^{\sqrt{\kappa}+\frac{1}{2}}}\sqrt{\frac{(2n+2\sqrt{\kappa}+1)\Gamma(n+2\sqrt{\kappa}+1)n!}{a\Gamma(n-2\kappa_1+1)\Gamma(n-2\kappa_2+1)}} \tag{3.35}$$

is defined from the orthogonality relation for Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ of the following form

$$\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta P_m^{(\alpha,\beta)}(x)\,P_n^{(\alpha,\beta)}(x)dx = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)n!}\delta_{mn},$$

within conditions $\alpha > -1$ and $\beta > -1$. Therefore, wavefunctions of the stationary states in the position representation are also orthogonal in the finite region $-\alpha < x < \alpha$:

$$\int_{-a}^{a}\left[\psi_m^{gGI}(x)\right]^*\psi_n^{gGI}(x)dx = \delta_{mn}.$$

Thanks to analytical expressions of energy spectrum (3.33) and wavefunctions (3.34) obtained via exact solution of the Schrödinger equation (3.9), we achieved our main goal. Now, we will discuss briefly some limit relations and special cases of these results in final section of the paper.

4. DISCUSSIONS AND LIMIT RELATIONS

Let's discuss obtained expressions for the energy spectrum (3.33) and wavefunctions of the stationary states (3.34) of a confined position-dependent mass harmonic oscillator.

It is clear that both energy spectrum (3.33) and wavefunctions of the stationary states (3.34) easily recover energy spectrum (3.5) and wavefunctions of the stationary states (3.6) in case of $g = 0$. They also recover energy spectrum (2.14) and wavefunctions of the stationary states (2.15) of the nonrelativistic quantum harmonic oscillator, if $a \to \infty$. It proves the correctness of our computations. Here, one needs to take into account two important results of the mathematics – first one is a Taylor expansion of the square root and second one special case relation between the Jacobi and Gegenbauer polynomials.

Our conclusion is that the model considered here is interesting and its behavior cordially differs from behavior of the ordinary harmonic oscillator under the external gravitational field. Confinement effects and unique non-linear picture of the energy spectrum appeared here can extend its potential applications in future.

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