COMMENT ON "A QUANTUM EXACTLY SOLVABLE NONLINEAR OSCILLATOR WITH QUASI-HARMONIC BEHAVIOR" AND "ALGEBRAIC SOLUTIONS OF SHAPE-INVARIANT POSITION-DEPENDENT EFFECTIVE MASS SYSTEMS" AND OTHERS

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Papers [3, 4] are devoted to the study of the quantum version of the nonlinear classical harmonic oscillator proposed in [1]. The authors of [3, 4] applied various quantization schemes for the classical Hamiltonian and expressed the wave functions of a quantum nonlinear harmonic oscillator in terms of Λ-dependent Hermite polynomials $\mathcal{H}_n(y,\Lambda)$ and $\tilde{\lambda}$- modified Hermite polynomials $\mathcal{H}_n(\zeta,\tilde{\lambda})$, respectively. We showed that these polynomials are not new, but in fact, for Λ<0 and $\tilde{\lambda}<0$, are Gegenbauer polynomials $C_n^\nu(x)$, and for $\Lambda>0$ and $\tilde{\lambda}>0$, they are special cases of pseudo-Jacobi polynomials $P_n(x;\nu,N)$ corresponding to the value of the parameter $\nu=0$. In addition, we have constructed a generating function for the polynomials $P_n(x;0,N)$ and established their connection with the polynomials $C_n^\nu(x)$, and also constructed two exactly solvable potentials associated with the pseudo-Jacobi polynomials.

Keywords: nonlinear harmonic oscillator, wave functions, Gegenbauer and pseudo-Jacobi polynomials, generating function, limit relations.

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1. INTRODUCTION

In [1], a nonlinear differential equation

$$(1+\lambda x^2)\ddot{x}-\lambda x\dot{x}^2+\alpha^2 x=0\,, \qquad (1.1)$$

describing a nonlinear classical harmonic oscillator was studied. Here λ and α are arbitrary real parameters, and the parameter λ characterizes the nonlinearity of the system. An important property of this equation is that it can be solved exactly. Its solution is a periodic function of time

$$x(t)=A\sin(\omega t+\phi), \qquad (1.2)$$

where ϕ is an arbitrary constant phase, and the oscillator frequency ω is related to the amplitude A as follows

$$\omega^2=\frac{\alpha^2}{1+\lambda A^2}\,. \qquad (1.3)$$

System (1.1) is described by a Lagrangian of the form

$$L=\frac{1}{2}\left(\frac{1}{1+\lambda x^2}\right)(\dot{x}^2-\alpha^2x^2) \qquad (1.4).$$

Consequently, the nonlinear system (1.1) can be regarded as a harmonic oscillator with a position-dependent mass

$$m(x)=\frac{1}{1+\lambda x^2}. \qquad (1.5)$$

The value $\lambda=0$ corresponds to a conventional linear harmonic oscillator with constant unit mass $m_0=1$. The classical Hamiltonian of the system (1.1) is equal to

$$H=\frac{1}{2}\,(1+\lambda x^2)p^2+\frac{1}{2}\left(\frac{\alpha^2x^2}{1+\lambda x^2}\right), \qquad (1.6)$$

where $p=\partial L/\partial\dot{x}=\dot{x}(1+\lambda x^2)$ is the momentum of the system. Note that for negative λ values, the mass function (1.5) has singularities at the points $x=\pm 1/\sqrt{|\lambda|}$, therefore, in this case, the system is analyzed in the interval $x\epsilon(-1/\sqrt{|\lambda|}\,,1/\sqrt{|\lambda|})$.

The quantum version of the nonlinear harmonic oscillator (1.1) was considered in [2], and papers [3, 4] are devoted to its further study. These authors applied various quantization schemes for the classical Hamiltonian (1.6). Here, a long-standing problem of the quantum model with a pozition-dependent mass is the way of ordering the ambiguities that appear due to the non-commutativity between the mass function $m(x)$ and the momentum operator $\hat{p}=-i\hbar\partial_x$ in the definition of the kinetic energy operator. A lot of work has been done in articles [3, 4] and important and interesting physical and mathematical results have been obtained.

The purpose of this commentary is to discuss and clarify some of these mathematical results. In this regard, consider the following Hamiltonians with position-dependent mass $m(x)$

$$\hat{H}=-\frac{1}{2m(x)}\partial_x^2-\frac{1}{\sqrt{2m(x)}}\left(\frac{1}{\sqrt{2m(x)}}\right)'\partial_x+V(x), \qquad (1.7)$$

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$$\hat{H} = -\frac{1}{2m(x)}\partial_x^2 - \left(\frac{1}{2m(x)}\right)'\partial_x + V(x), \tag{1.8}$$

where $V(x)$ is the potential energy, and the prime denotes the derivative with respect to x.

Hamiltonians (1.7) and (1.8) are Hermitian (under certain boundary conditions depending on the form of the mass function $m(x)$), respectively, with respect to the scalar products

$$\int_a^b \psi^*(x)\phi(x)\sqrt{m(x)}\,dx \text{ and } \int_a^b \psi^*(x)\phi(x)dx, \tag{1.9}$$

where the interval (a, b) can be finite or infinite.

2. BRIEF DISCUSSION OF THE RESULTS OF PAPERS [3, 4]

2.1. Let us now discuss separately and briefly the results of [3, 4]. In [3], a Hamiltonian of the form (1.7) with the mass function (1.5) was used

$$\hat{H} = \frac{\hbar^2}{2m_0}\left[-(1+\lambda x^2)\partial_x^2 - \lambda x\partial_x\right] + \frac{1}{2}\left(\frac{g\,x^2}{1+\lambda x^2}\right), \tag{2.1}$$

where $g = m_0\alpha^2 + \lambda\hbar\alpha$ and $m_0 = \text{const}$, and the parameter λ can take positive and negative values. Operator (2.1) is Hermitian in the space $L^2(R, d\rho)$ for $\lambda > 0$ and in the space $L^2((-1/\sqrt{|\lambda|}, 1/\sqrt{|\lambda|}), d\rho)$ for $\lambda < 0$, where $d\rho = (1+\lambda x^2)^{-1/2}\,dx$. In this paper, the Schrödinger equation with the Hamiltonian (2.1) is written in the form

$$\left[(1+\Lambda y^2)\partial_y^2 + \Lambda y\partial_y - \frac{(1+\Lambda)y^2}{1+\Lambda y^2} + 2e\right]\psi = 0, \tag{2.2}$$

where $y = \sqrt{m_0\alpha/\hbar}\;x$, $\Lambda = \frac{\hbar}{m_0\alpha}\lambda$ and $e = E/\hbar\alpha$ are dimensionless quantities. The wave functions are sought in the form $\psi(y,\Lambda) = h(y,\Lambda)\,(1+\Lambda y^2)^{-1/(2\Lambda)}$, where the functions $h = h(y,\Lambda)$ satisfy the second order differential equation

$$(1+\Lambda y^2)h'' + (\Lambda - 2)yh' + (2e-1)h = 0. \tag{2.3}$$

Polynomial solutions of the equation (2.3) $h = \mathcal{H}_n(y,\Lambda)$ in [3] were called Λ - dependent Hemite polynomials and it was proved that they form an orthogonal system. For them, the Rodrigues formula is obtained

$$\mathcal{H}_n(y,\Lambda) = (-1)^n\,(1+\Lambda y^2)^{1/\Lambda+1/2}\,\frac{d^n}{dy^n}\left[(1+\Lambda y^2)^{n-1/\Lambda-1/2}\right], \tag{2.4}$$

where $n = 0,1,2,3,\ldots$

Thus, the wave functions of the bound states of the Λ-deformed nonlinear quantum oscillator (2.2) have the form

$$\psi_n(y,\Lambda) = \mathcal{H}_n(y,\Lambda)(1+\Lambda y^2)^{-1/(2\Lambda)},\; n = 0,1,2,3,\ldots \tag{2.5}$$

The energy levels corresponding to these wave functions are not equidistant and equal to

$$e_n = \left(n+\frac{1}{2}\right) - \frac{1}{2}n^2\,\Lambda. \tag{2.6}$$

It was shown in [3] that in the case when $\Lambda > 0$, number of energy levels e_n is finite, i.e. $n = 0,1,2,3,\ldots, N_\Lambda$, where N_Λ is the largest integer not exceeding $1/\Lambda$, and in the case when $\Lambda < 0$, it is infinite. In the limit $\Lambda \to 0$, expressions (2.5) and (2.6) coincide with the corresponding expressions for the linear nonrelativistic harmonic oscillator

$$\lim_{\Lambda\to 0}\psi_n(y,\Lambda) = H_n(y)e^{-(1/2)y^2} \quad\text{and}\quad \lim_{\Lambda\to 0} e_n = n + \frac{1}{2}, \tag{2.7}$$

where $H_n(y)$ are Hermite polynomials. In addition, in [3], an expression for the generating function for other Λ-dependent Hermite polynomials $\widetilde{\mathcal{H}}_n(y,\Lambda)$ is given:

$$(1+\Lambda(2ty - t^2))^{1/\Lambda} = \sum_{n=0}^{\infty}\widetilde{\mathcal{H}}_n(y,\Lambda)\,\frac{t^n}{n!}. \tag{2.8}$$

The polynomials $\mathcal{H}_n(y,\Lambda)$ and $\widetilde{\mathcal{H}}_n(y,\Lambda)$ coincide in the fundamental part (i.e., in the y -dependent polynomial part) and differ from each other only by constant factors depending on the number n.

2.2. Let us now consider the results of paper [4]. In it, the quantum analogue of the classical nonlinear harmonic oscillator (1.1) was described by a Hamiltonian of the form (1.8), in which the mass function $m(x)$ was taken in the following three forms:

$$m(x) = \frac{1}{1+\lambda x^2}\,,\ m(x) = \ (1+\frac{x^2}{\lambda})^{-1},\ m(x) = \frac{2}{1-(\lambda x)^2}. \tag{2.9}$$

Let us briefly discuss each of these cases separately.

Case 1: $\boldsymbol{m(x) = (1+\lambda x^2)^{-1}}$**.** In this case, Hamiltonian (1.8) is equal to

$$\widehat{H} = \frac{1}{2}\Big[-(1+\lambda x^2)\partial_x^2 - 2\lambda x\partial_x + \frac{\alpha^2x^2}{1+\lambda x^2}\Big]. \tag{2.10}$$

Hamiltonian (2.10) is Hermitian in the space $L^2(R, dx)$ for $\lambda > 0$ and in the space $L^2((-1/\sqrt{|\lambda|}\,, 1/\sqrt{|\lambda|}\,), dx)$ for $\lambda < 0$. It was shown in [4] that the eigenfunctions and eigenvalues of this Hamiltonian in terms of the dimensionless quantities $\zeta = \sqrt{\alpha}\,x$ и $\tilde{\lambda} = \lambda/\alpha$ have the form

$$\varphi_n(\zeta,\tilde{\lambda}) = \mathcal{H}_n(\zeta,\tilde{\lambda})\,(1+\tilde{\lambda}\,\zeta^2)^{-1/(2\tilde{\lambda})}, \tag{2.11}$$

$$E_n = \alpha\left[\left(n+\frac{1}{2}\right) - \tilde{\lambda}\left(\frac{n(n+1)}{2}\right)\right], n = 0,1,2,3,..., \tag{2.12}$$

where $\mathcal{H}_n$ were called $\tilde{\lambda}$-modified Hermite polynomials. They are given by the Rodrigues formula

$$\mathcal{H}_n(\zeta,\tilde{\lambda}) = (-1)^n[(1+\tilde{\lambda}\,\zeta^2)^{1/\tilde{\lambda}}\frac{d^n}{d\zeta^n}\,(1+\tilde{\lambda}\,\zeta^2)^{n-1/\tilde{\lambda}}], n = 0,1,2,3, ... \tag{2.13}$$

Moreover, in [4], for several other $\tilde{\lambda}$-modified Hermite polynomials $\widetilde{\mathcal{H}}_n(\zeta,\tilde{\lambda})$, the following generating function was written

$$[(1+\tilde{\lambda}\,(2t\zeta - t^2)]^{-1/2+1/\tilde{\lambda}} = \sum_{n=0}^{\infty}\widetilde{\mathcal{H}}_n(\zeta,\tilde{\lambda})\frac{t^n}{n!}\,. \tag{2.14}$$

The polynomials $\mathcal{H}_n(\zeta,\tilde{\lambda})$ and $\widetilde{\mathcal{H}}_n(\zeta,\tilde{\lambda})$, as in [3], differ from each other by constant factors depending on the number n. It should also be noted that the wave functions (2.11) $\varphi_n(\zeta,\tilde{\lambda}) \equiv \varphi_n(x)$ for $\lambda > 0$ are not square integrable in the space $L^2(R, dx)$, i.e.$\int_{-\infty}^{\infty}|\varphi_n(x)|^2dx = \infty$ for all $n > 1/\tilde{\lambda}$.

It was shown in [4] that wave functions (2.11) and energy levels (2.12) at the harmonic limit $\lambda \to 0$ reproduce the results of a nonrelativistic harmonic oscillator with constant mass.

Case 2: $\boldsymbol{m(x) = \left(1+\frac{x^2}{\lambda}\right)^{-1}}$**.** In this case, the wave functions, the energy spectrum can be obtained by replacing $\lambda \to \lambda^{-1}$ from the corresponding expressions for the case 1.

Case 3: $\boldsymbol{m(x) = \frac{2}{1-(\lambda x)^2}}$**.** In this case, the Hamiltonian of the nonlinear oscillator (1.8), regardless of the sign λ, is Hermitian only in space $L^2\big((-1/|\lambda|\,, 1/|\lambda|), dx\big)$. Its eigenfunctions and eigenvalues are also can be obtained from the results of the case 1 for $\lambda < 0$. Therefore, we will only discuss the results of the case 1.

The purpose of this paper is to show that Λ-dependent Hermite polynomials $\mathcal{H}_n(y,\Lambda)$ (2.4) and $\widetilde{\mathcal{H}}_n(y,\Lambda)$ (2.8), as well as the modified Hermite polynomials $\mathcal{H}_n(\zeta,\tilde{\lambda})$ (2.13) and $\widetilde{\mathcal{H}}_n(\zeta,\tilde{\lambda})$ (2.14), introduced respectively in [3] and [4], are not new polynomials, but are, for $\Lambda < 0$ and $\tilde{\lambda} < 0$, the Gegenbauer polynomials $C_n^{\nu}(x)$, $n = 0,1,2,3,...$, and for $\Lambda > 0$ and $\tilde{\lambda} > 0$, they are special cases of pseudo-Jacobi polynomials $P_n(x;\nu,N), n = 0,1,2,3,...N$, corresponding to the value of the parameter $\nu = 0$. (Compare with the results of papers [5 - 7].)

In addition, in this paper we will construct a generating function for the polynomials $P_n(x;0,N)$ and establish their connection with the polynomials $C_n^{\nu}(x)$. Our other goal is to construct exactly solvable potentials associated with the pseudo-Jacobi polynomials.

The organization of the paper is as follows. Section 2 briefly discusses results obtained in [3, 4].) In Section 3, some basic properties of Hermite, Gegenbauer and pseudo-Jacobi polynomials are recalled. These properties include their hypergeometric expressions, orthogonality and recurrence relations, and differential equations. They are used throughout the main text. Section 4 is devoted to obtaining the main results of this work. Conclusion is presented in Section 5. In Appendices 1 and 2, we study two limit relations concerning the properties of orthogonal polynomials. Appendix 3 is devoted to the construction of exactly solvable potentials associated with pseudo-Jacobi polynomials.

3. BASIC PROPERTIES OF HERMITE, GEGENBAUER AND PSEUDO-JACOBI POLYNOMIALS

In this section, we give the main formulas for Hermite, Gegenbauer and pseudo-Jacobi polynomials. All of these can be found in [8,9], but it is convenient to list them here for further reference.

Hermite polynomials are defined in terms of ${}_2F_0$ hypergeometric functions as follows:

$$H_n(x) = (2x)^n\,{}_2F_0\left(-\frac{n}{2},\ -\frac{(n-1)}{2}\middle|\ -\frac{1}{x^2}\right). \tag{3.1}$$

They are exact solutions of the following second-order differential equation

$$y''(x) - 2xy'(x) + 2ny(x) = 0, \quad y(x) = H_n(x). \tag{3.2}$$

Hermite polynomials satisfy an orthogonality relation on the interval $(-\infty, \infty)$:

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx = d_{\mathrm{H}n}^2 \delta_{mn} \qquad d_{\mathrm{H}n}^2 = 2^n n! \sqrt{\pi} \tag{3.3}$$

and a recurrence relation of the form

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x). \tag{3.4}$$

We also write for them the Rodrigues formula and the generating function

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \; n = 0,1,2, \dots , \tag{3.5}$$

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}. \tag{3.6}$$

Gegenbauer polynomials $C_n^{\nu}(x)$ are defined in terms of ${}_2F_1$ hypergeometric function as follows:

$$C_n^{\nu}(x) = \frac{(2\nu)_n}{n!} \; {}_2F_1\left(\begin{matrix} -n, & n+2\nu \\ & \nu + \frac{1}{2} \end{matrix} \middle| \frac{1-x}{2} \right), \; \nu \neq 0 \tag{3.7}$$

and are exact solutions of the second-order differential equation

$$(1-x^2)y''(x) - (2\nu+1)xy'(x) + n(n+2\nu)y(x) = 0 \,, \; y(x) = C_n^{\nu}(x). \tag{3.8}$$

They satisfy an orthogonality condition on the interval (-1,1):

$$\int_{-1}^{1} (1-x^2)^{\nu-\frac{1}{2}} \, C_m^{\nu}(x) C_n^{\nu}(x) dx = d_{\mathrm{C}n}^2 \delta_{mn},$$

$$d_{\mathrm{C}n}^2 = \frac{\pi\Gamma(n+2\nu)2^{1-2\nu}}{\{\Gamma(\nu)\}^2 (n+\nu) n!}, \quad \nu > -\frac{1}{2} \;\text{ and }\; \nu \neq 0. \tag{3.9}$$

We also write down the Rodrigues formula for them and the generating function

$$C_n^{\nu}(x) = (-1)^n \frac{(2\nu)_n}{(\nu+\frac{1}{2})_n \, 2^n \, n!} (1-x^2)^{-\nu+\frac{1}{2}} \frac{d^n}{dx^n} [(1-x^2)^{n+\nu-\frac{1}{2}}], \; n = 0,1,2,3 \dots, \tag{3.10}$$

$$(1-2xt+t^2)^{-\nu} = \sum_{n=0}^{\infty} C_n^{\nu}(x) t^n \,. \tag{3.11}$$

Pseudo-Jacobi polynomials $P_n(x; \nu, N)$ are defined in terms of ${}_2F_1$ hypergeometric functions as follows:

$$P_n(x;\nu,N) = \frac{(-2i)^n \, (-N+i\nu)_n}{(n-2N-1)_n} \; {}_2F_1\left(\begin{matrix} -n, & n-2N-1 \\ & -N+i\nu \end{matrix} \middle| \frac{1-ix}{2} \right), \quad n = 0,1,2, \dots, N. \tag{3.12}$$

Herein, ν is an arbitrary real parameter and N is an arbitrary positive integer, т. е. $N = 1,2,3, \dots$ The polynomials $P_n(x;\nu,N)$ are real polynomials in x of degree n, and n is restricted by N. They are also the exact solution of a second-order differential equation, namely

$$(1+x^2)y''(x) + 2(\nu - Nx)y'(x) - n(n-2N-1)y(x) = 0, \quad y(x) = P_n(x;\nu,N) \tag{3.13}$$

and also satisfy an orthogonality relation on the interval $(-\infty, \infty)$:

$$\int_{-\infty}^{\infty} (1+x^2)^{-N-1} e^{2\nu \arctan x} P_m(x;\nu,N) \, P_n(x;\nu,N) dx = d_{\mathrm{N}n}^2(\nu) \delta_{nm},$$

$$d_{\mathrm{N}n}^2(\nu) = \pi n! \frac{\Gamma(2N+1-2n)\Gamma(2N+2-2n)2^{2n-2N}}{\Gamma(2N+2-n)|\Gamma(N+1-n+i\nu)|^2}. \tag{3.14}$$

We also write down for pseudo-Jacobi polynomials a Rodrigues-type formula

$$P_n(x;\nu,N) = \frac{(1+x^2)^{N+1} e^{-2\nu \arctan x}}{(n-2N-1)_n} \frac{d^n}{dx^n} [(1+x^2)^{n-N-1} e^{2\nu \arctan x}] \tag{3.15}$$

and a recurrence relation

$$P_{n+1}(x;\nu,N) = A_n P_n(x;\nu,N) + B_n P_{n-1}(x;\nu,N),$$

$$A_n(x,\nu) = x - \frac{\nu(N+1)}{(n-N-1)(n-N)},\ B_n(\nu) = \frac{n(n-2N-2)(n-N-1-i\nu)(n-N-1+i\nu)}{(2n-2N-3)(n-N-1)^2(2n-2N-1)}. \quad (3.16)$$

We emphasize that the following limit relations hold between the Gegenbauer and Hermite polynomials, as well as between the pseudo-Jacobi and Hermite polynomials

$$\lim_{\nu\to\infty} \nu^{-\frac{n}{2}}\ C_n^{\nu+\frac{1}{2}}\left(\frac{x}{\sqrt{\nu}}\right) = \frac{1}{n!} H_n(x), \quad (3.17)$$

$$\lim_{N\to\infty} N^{\frac{n}{2}} P_n\left(\frac{x}{\sqrt{N}};\ \nu, N\right) = \frac{1}{2^n} H_n(x), \quad (3.18)$$

Relation (3.17) is given in [6], and relation (3.18) can be proved by the method of mathematical induction.

4. MAİN RESULTS

In this section, we present the main results of our work. Consider first the polynomials defined by formulas (2.4) and (2.8) and introduced in [3].

1) Let be $\Lambda < 0$. Comparing of the Rodrigues formula for Λ-dependent polynomials $\mathcal{H}_n(y,\Lambda)$ (2.4) with Rodrigues formula for the Gegenbauer polynomials $C_n^\nu(x)$ (3.10), we obtain a formula that expresses the polynomials $\mathcal{H}_n(y,\Lambda)$ in terms of the Gegenbauer polynomials $C_n^\nu(x)$

$$\mathcal{H}_n(y,\Lambda) = 2^n\, n!\, \frac{\left(\frac{1}{|\Lambda|}+\frac{1}{2}\right)_n}{\left(\frac{2}{|\Lambda|}\right)_n}\ (|\Lambda|)^{n/2}\ C_n^{1/|\Lambda|}\left(\sqrt{|\Lambda|}y\right),\ \ \Lambda < 0. \quad (4.1)$$

2) Let be $\Lambda > 0$. Comparison of the Rodrigues formula (2.4) with Rodrigues formula for the pseudo-Jacobi polynomials $P_n(x;0,N)$ (3.15) gives us a relation connecting the polynomials $\mathcal{H}_n(y,\Lambda)$ in terms of the pseudo-Jacobi polynomials $P_n(x;0,N)$, i.e.

$$\mathcal{H}_n(y,\Lambda) = (-1)^n\ \left(n-\frac{2}{\Lambda}\right)_n\ \Lambda^{n/2}\ P_n\left(\sqrt{\Lambda}y; 0, \frac{1}{\Lambda}-\frac{1}{2}\right),\ \ \Lambda > 0. \quad (4.2)$$

3) Let be $\Lambda < 0$. Let us now compare the generating function for other Λ-dependent polynomials $\widetilde{\mathcal{H}}_n(y,\Lambda)$ (2.8) with a generating function for the Gegenbauer polynomials $C_n^\nu(x)$ (3.11). Then we obtain the relation between the $\widetilde{\mathcal{H}}_n(y,\Lambda)$ polynomials and the Gegenbauer polynomials

$$\widetilde{\mathcal{H}}_n(y,\Lambda) = |\Lambda|^{\frac{n}{2}}\, n!\, C_n^{\frac{1}{|\Lambda|}}\left(\sqrt{|\Lambda|}y\right),\ \Lambda < 0. \quad (4.3)$$

4) Let be $\Lambda > 0$. To find the explicit form of the Λ-dependent polynomials $\widetilde{\mathcal{H}}_n(y,\Lambda)$ for $\Lambda > 0$ we will proceed as follows. Analysis of the properties of polynomials $\widetilde{\mathcal{H}}_n(y,\Lambda)$ by formula (2.8) allows us to conclude that these polynomials are proportional to the pseudo-Jacobi polynomials $P_n\left(\sqrt{\Lambda}y; 0, \frac{1}{\Lambda}-\frac{1}{2}\right)$, i.e.

$$\widetilde{\mathcal{H}}_n(y,\Lambda) = c_n P_n\left(\sqrt{\Lambda}y; 0, \frac{1}{\Lambda}-\frac{1}{2}\right),\ n = 0, 1, 2, 3, \ldots, \frac{1}{\Lambda}+\frac{1}{2}. \quad (4.4)$$

Here c_n are some coefficients. To find these coefficients, we compare the recurrence relation for the polynomials $\widetilde{\mathcal{H}}_n(y,\Lambda)$, obtained in [3], with the recurrence relation for the polynomials $P_n(\sqrt{\Lambda}y; 0, \frac{1}{\Lambda}-\frac{1}{2})$ (3.16). These recurrence relations are of the form

$$\widetilde{\mathcal{H}}_{n+1} = 2y(1-n\Lambda)\widetilde{\mathcal{H}}_n - n[2-(n-1)\Lambda]\widetilde{\mathcal{H}}_{n-1}, \quad (4.5)$$

$$P_{n+1} = \sqrt{\Lambda}yP_n + \frac{n(n-1-\frac{2}{\Lambda})}{(2n-2-\frac{2}{\Lambda})(2n-\frac{2}{\Lambda})}P_{n-1}, \quad (4.6)$$

where $n \geq 1$. Formulas (4.4) - (4.6) lead to a simple recurrence relation for the coefficients c_n as $c_{n+1} = \frac{2}{\sqrt{\Lambda}}(1-n\Lambda)c_n$. It has the following solution

$$c_n(\Lambda) = \left(-2\sqrt{\Lambda}\right)^n\left(-\frac{1}{\Lambda}\right)_n. \quad (4.7)$$

Thus, according to (4.4) and (4.7) for $\Lambda > 0$ polynomials $\widetilde{\mathcal{H}}_n(y,\Lambda)$ up to constant coefficients coincide with the pseudo-Jacobi polynomials $P_n(\sqrt{\Lambda}y; 0, \frac{1}{\Lambda} - \frac{1}{2})$:

$$\widetilde{\mathcal{H}}_n(y,\Lambda) = \left(-2\sqrt{\Lambda}\right)^n \left(-\frac{1}{\Lambda}\right)_n P_n\left(\sqrt{\Lambda}y; 0, \frac{1}{\Lambda} - \frac{1}{2}\right), \ \Lambda > 0. \tag{4.8}$$

Let us now make two remarks:

a) according to the definition of the pseudo-Jacobi polynomials (3.12), in formulas (4.2) and (4.8) the expression $\frac{1}{\Lambda} - \frac{1}{2}$ is equal to a positive integer, i.e. $\frac{1}{\Lambda} - \frac{1}{2} = 1, 2, 3, ...$, and $n = 0, 1,2,3, ..., \frac{1}{\Lambda} - \frac{1}{2}$. Therefore, as was noted in [3], for $\Lambda > 0$ the number of energy levels of the nonlinear oscillator (2.2) is bounded. It is equal to $\frac{1}{\Lambda} + \frac{1}{2}$;

b) the ratio $h_n(\Lambda) = \mathcal{H}_n(y,\Lambda)/\widetilde{\mathcal{H}}_n(y,\Lambda)$ of the polynomials (4.1) and (4.3), as well as of the polynomials (4.2) and (4.8) are equal to

$$h_n(\Lambda < 0) = 2^n \left(\frac{1}{2} - \frac{1}{\Lambda}\right)_n \Big/ \left(-\frac{2}{\Lambda}\right)_n ,$$

$$h_n(\Lambda > 0) = \left(n - \frac{2}{\Lambda}\right)_n / [2^n \left(-\frac{1}{\Lambda}\right)_n], \tag{4.9}$$

respectively. If we ignore the sign of the parameter Λ , then $h_n(\Lambda < 0)$ and $h_n(\Lambda > 0)$ will coincide.

5) Since we have generating function (2.8) for the polynomials $\widetilde{\mathcal{H}}_n(y,\Lambda)$, using relation (4.8), which is valid for $\Lambda > 0$, we can obtain the generating function for the pseudo-Jacobi polynomials $P_n(x; 0, N)$. It has the form

$$\left(1 + 2\sqrt{\frac{2}{2N+1}}tx - \frac{2}{2N+1}t^2\right)^{N+\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\left(-2\sqrt{\frac{2}{2N+1}}\right)^n \left(-\frac{2N+1}{2}\right)_n P_n(x; 0, N). \tag{4.10}$$

If in (4.10) we replace x on $x/\sqrt{N}$ and go to the limit $N \to \infty$, then we obtain, as expected, the generating function for the Hermite polynomials $H_n(x)$ (3.6) (the proof is given in Appendix 1).

6) Consider now the polynomials (2.13) and (2.14) [4]. Comparison of the Rodrigues formula for the polynomials $\mathcal{H}_n(\zeta,\tilde{\lambda})$ (2.13) for $\tilde{\lambda} < 0$ with Rodrigues formula for Gegenbauer polynomials $C_n^{\nu}(x)$ (3.10), and for $\tilde{\lambda} > 0$ with Rodrigues formula for pseudo-Jacobi polynomials $P_n(x; 0, N)$ (3.15) gives the following relations between these polynomials

$$\mathcal{H}_n(\zeta,\tilde{\lambda}) = 2^n\, n! \frac{\left(1+\frac{1}{|\tilde{\lambda}|}\right)_n}{\left(1+\frac{2}{|\tilde{\lambda}|}\right)_n} \,(|\tilde{\lambda}|)^{n/2}\, C_n^{1/2+1/|\tilde{\lambda}|}\left(\sqrt{|\tilde{\lambda}|}\zeta\right), \quad \tilde{\lambda} < 0, \tag{4.11}$$

$$\mathcal{H}_n(\zeta,\tilde{\lambda}) = (-1)^n \left(n + 1 - \frac{2}{\tilde{\lambda}}\right)_n (\tilde{\lambda})^{n/2} P_n\left(\sqrt{\tilde{\lambda}}\,\zeta; 0, \frac{1}{\tilde{\lambda}} - 1\right), \quad \tilde{\lambda} > 0. \tag{4.12}$$

7) To find the connection between the polynomials $\widetilde{\mathcal{H}}_n(\zeta,\tilde{\lambda})$ (2.14) with Gegenbauer polynomials $C_n^{\nu}(x)$ (3.10) and pseudo-Jacobi polynomials $P_n(x; 0, N)$ (3.15) we proceed as follows. Let us compare the generating function for $\widetilde{\mathcal{H}}_n(\zeta,\tilde{\lambda})$ (2.14) for $\tilde{\lambda} < 0$ with a generating function for $C_n^{\nu}(x)$ (3.11), and for $\tilde{\lambda} > 0$ with a generating function for $P_n(x; 0, N)$ (4.10). We get

$$\widetilde{\mathcal{H}}_n(\zeta,\tilde{\lambda}) = \text{n!}\ |\tilde{\lambda}|^{\frac{n}{2}}\, C_n^{\frac{1}{|\tilde{\lambda}|}+\frac{1}{2}}(\sqrt{|\tilde{\lambda}|}\zeta), \quad \tilde{\lambda} < 0, \tag{4.13}$$

$$\widetilde{\mathcal{H}}_n(\zeta,\tilde{\lambda}) = (-2\sqrt{\tilde{\lambda}})^n(\tfrac{1}{2} - \tfrac{1}{\tilde{\lambda}})_n\, P_n\left(\sqrt{\tilde{\lambda}}\zeta; 0, \frac{1}{\tilde{\lambda}} - 1\right), \quad \tilde{\lambda} > 0. \tag{4.14}$$

In formulas (4.12) and (4.14), the parameter $\tilde{\lambda}$ ranges over discrete values $\tilde{\lambda} = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$, and the values of the number n are bounded from above, i.e. $n = 0, 1,2,3, ... , 1/\tilde{\lambda} - 1$. Thus, like the system (2.2), the system described by Hamiltonian (2.10), for $\tilde{\lambda} > 0$ also has a finite number of levels equal to $1/\tilde{\lambda}$.

Similarly to (4.9), we now find the ratio $h_n(\tilde{\lambda}) = \mathcal{H}_n(\zeta,\tilde{\lambda})/\widetilde{\mathcal{H}}_n(\zeta,\tilde{\lambda})$ of the polynomials (4.11) and (4.13), as well as of the polynomials (4.12) and (4.14). They are equal to

$$h_n(\tilde{\lambda} < 0) = 2^n \frac{\left(1-\frac{1}{\tilde{\lambda}}\right)_n}{\left(1-\frac{2}{\tilde{\lambda}}\right)_n}, \quad h_n(\tilde{\lambda} > 0) = \left(n + 1 - \frac{2}{\tilde{\lambda}}\right)_n / [2^{\,n}(\tfrac{1}{2} - \tfrac{1}{\tilde{\lambda}})_n], \tag{4.15}$$

respectively. Here, as in the case (4.9), if we ignore the sign of the parameter $\tilde{\lambda}$, then $h_n(\tilde{\lambda} < 0)$ and $h_n(\tilde{\lambda} > 0)$ will coincide.

8) We can now write out the explicit form of the orthonormal wave functions $\psi_n(y,\Lambda)$ (2.5) and $\varphi_n(\zeta,\tilde{\lambda})$ (2.11). The normalized wave functions have the following form: 1) for $\Lambda<0$ and $\tilde{\lambda}<0$

$$\psi_n(y,\Lambda)=2^{-1/\Lambda}\,\Gamma\left(-\frac{1}{\Lambda}\right)\left(\frac{\sqrt{-\Lambda}\ \mathrm{n}!\left(n-\frac{1}{\Lambda}\right)}{2\pi\,\Gamma(n-\frac{2}{\Lambda})}\right)^{1/2}(1+\Lambda y^2)^{-1/(2\Lambda)-1/4}C_n^{-1/\Lambda}\left(\sqrt{-\Lambda}y\right), \tag{4.16}$$

$$\varphi_n(\zeta,\tilde{\lambda})=2^{-1/\tilde{\lambda}}\,\Gamma\left(-\frac{1}{\tilde{\lambda}}+\frac{1}{2}\right)\left(\frac{\sqrt{-\tilde{\lambda}}\,\mathrm{n}!\,(n+\frac{1}{2}-\frac{1}{\tilde{\lambda}})}{\pi\,\Gamma!(n+1-\frac{2}{\tilde{\lambda}})}\right)^{1/2}\left(1+\tilde{\lambda}\zeta^2\right)^{-1/(2\tilde{\lambda})}C_n^{-1/\tilde{\lambda}+1/2}\left(\sqrt{-\tilde{\lambda}}\zeta\right); \tag{4.17}$$

2) for $\Lambda>0$ and $\tilde{\lambda}>0$

$$\psi_n(y,\Lambda)=2^{1/\Lambda-1-n}\frac{\Gamma\left(\frac{1}{\Lambda}+\frac{1}{2}-n\right)}{\Gamma\left(\frac{2}{\Lambda}-2n\right)}\left(\frac{\sqrt{\Lambda}\ \Gamma\left(\frac{2}{\Lambda}+1-2n\right)}{\pi\,n!\left(\frac{1}{\Lambda}-n\right)}\right)^{1/2}(1+\Lambda y^2)^{-1/(2\Lambda)-1/4}\,P_n\left(\sqrt{\Lambda}y;0,\frac{1}{\Lambda}-\frac{1}{2}\right), \tag{4.18}$$

$$\varphi_n(\zeta,\tilde{\lambda})=2^{1/\tilde{\lambda}-1-n}\frac{\Gamma(\frac{1}{\tilde{\lambda}}-n)}{\Gamma(\frac{2}{\tilde{\lambda}}-1-2n)}\left(\frac{\sqrt{\tilde{\lambda}}\ \Gamma(\frac{2}{\tilde{\lambda}}-n)}{\pi\,n!(\frac{2}{\tilde{\lambda}}-1-2n)}\right)^{1/2}\left(1+\tilde{\lambda}\zeta^2\right)^{-1/(2\tilde{\lambda})}\,P_n\left(\sqrt{\tilde{\lambda}}\zeta;0,\frac{1}{\tilde{\lambda}}-1\right). \tag{4.19}$$

These wave functions are normalized by the conditions: 1) for $\Lambda<0$ and $\tilde{\lambda}<0$

$$\int_{-a_\Lambda}^{a_\Lambda}\psi_n^*(y,\Lambda)\psi_m(y,\Lambda)dy=\int_{-a_\lambda}^{a_\lambda}\varphi_n^*(\zeta,\tilde{\lambda})\varphi_m(\zeta,\tilde{\lambda})d\zeta=\delta_{nm}, \tag{4.20}$$

2) for $\Lambda>0$ and $\tilde{\lambda}>0$

$$\int_{-\infty}^{\infty}\psi_n^*(y,\Lambda)\psi_m(y,\Lambda)dy=\int_{-\infty}^{\infty}\varphi_n^*(\zeta,\tilde{\lambda})\varphi_m(\zeta,\tilde{\lambda})d\zeta=\delta_{nm}, \tag{4.21}$$

where $a_\Lambda=(|\Lambda|)^{-1/2}$ and $a_\lambda=\left(\alpha/|\tilde{\lambda}|\right)^{1/2}$. In calculating them, we used the orthogonality conditions for the Gegenbauer (3.9) and pseudo-Jacobi polynomials (3.14).

9) We can also establish a connection between the Gegenbauer polynomials $C_n^\nu(x)$ (3.7) and pseudo-Jacobi polynomials $P_n(x;0,N)$ (3.12). To do this, we first discuss some of the results of [10]. In this paper, the quantum version of the nonlinear classical harmonic oscillator (1.1) was also investigated. In this case, it is described by the Schrödinger equation

$$\left[(1-\lambda x^2)\partial_x^2+2a\lambda x\partial_x+b+\frac{cx^2}{1-\lambda x^2}\right]\psi=0, \tag{4.22}$$

where a,b,c are some constant parameters, and $c<0$. Their values in [10] defined as

$$a=\bar{\gamma}-\bar{\alpha}-1,\,,b=2\lambda\bar{\gamma}+\frac{2E}{\hbar^2},\,c=-(2\overline{\alpha\gamma}\,\lambda^2+\frac{k}{\hbar^2}). \tag{4.23}$$

In the case $\lambda>0$ the solution to equation (4.22) was analyzed in detail in [10]. The case $\lambda<0$ requires additional analysis. In this case, the eigenfunctions of the equation (4.22) in [10] are expressed in terms of the Gegenbauer polynomials as follows

$$\psi_n(x)=N_n(1+|\lambda|x^2)^{\frac{a+1}{2}}P_{n-\mu}^{\mu}\left(i\sqrt{|\lambda|}x\right), \tag{4.24}$$

$$P_{n-\mu}^{\mu}(ix)=\frac{(-1)^{\frac{n-\mu}{2}}2^{n-\mu}n!\Gamma\left(n-\mu+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(1+2n-2\mu)}(1+x^2)^{\frac{n-\mu}{2}}C_n^{\mu-n}\left(\frac{x}{\sqrt{1+x^2}}\right). \tag{4.25}$$

Here $P_\nu^\mu(z)$ are Legendre functions of the first kind [11]

$$P_\nu^\mu(z)=\frac{2^\mu(z^2-1)^{-\frac{\mu}{2}}}{\Gamma(1-\mu)}\ {}_2F_1\left(\begin{matrix}-\nu-\mu, & 1+\nu-\mu\\ & 1-\mu\end{matrix}\middle|\frac{1-z}{2}\right). \tag{4.26}$$

Hence, if we put $\nu+\mu=n=0,1,2,3,\ldots$ in (4.26) then the hypergeometric function will be a polynomial and we get

$$P_{n-\mu}^{\mu}(ix)=\frac{2^\mu(-1)^{-\frac{\mu}{2}}}{\Gamma(1-\mu)}(1+x^2)^{-\frac{\mu}{2}}\ {}_2F_1\left(\begin{matrix}-n, & n+1-2\mu\\ & 1-\mu\end{matrix}\middle|\frac{1-ix}{2}\right). \tag{4.27}$$

On the other hand, in Appendix 2 we show that in the case $\lambda < 0$ the eigenfunctions of equation (4.22) are expressed in terms of the pseudo-Jacobi polynomials $P_n(x; 0, N)$. Therefore, there must be a connection between the polynomials $P_n(x; 0, N)$ and $C_n^{\nu}(x)$. Let's find this connection. From equalities (3.12) and (4.27) it follows:

$$P_{n-\mu}^{\mu}(ix) = \frac{2^{\mu}(-1)^{-\frac{\mu}{2}}(n+1-2\mu)_n}{\Gamma(1-\mu)(-2i)^n(1-\mu)_n}(1+x^2)^{-\frac{\mu}{2}}P_n(x; 0, \mu-1). \quad (4.28)$$

Equating now the expressions contained in formulas (4.25) and (4.28), we find the desired connection

$$P_n(x; 0, N) = (-1)^n n!\, 2^{-2n} \frac{(-2N-1)_n}{\left(-N-\frac{1}{2}\right)_n (-N)_n}(1+x^2)^{\frac{n}{2}} C_n^{N+1-n}\left(\frac{x}{\sqrt{1+x^2}}\right). \quad (4.29)$$

This shows that the orthogonality condition for the pseudo-Jacobi polynomials (3.14) for $\nu = 0$ reduces to the orthogonality condition for the Gegenbauer polynomials (3.9).

5. CONCLUSION

In this paper, for clarification, we discussed some of the mathematical results obtained in [3] and [4]. In these papers, a quantum version of an exactly solvable one-dimensional nonlinear classical harmonic oscillator which was initially considered by Mathews and Lakshmanan [1] was studied. These authors have applied various quantization schemes. The wave functions of the considered nonlinear model of the harmonic oscillator in [3] were expressed in terms of Λ - dependent Hermite polynomials $\mathcal{H}_n(y, \Lambda)$, and in [4] in terms $\tilde{\lambda}$ - dependent Hermite polynomials $\mathcal{H}_n(\zeta, \tilde{\lambda})$, where parameter Λ (or $\tilde{\lambda}$) characterizes the nonlinearity of the oscillator. The authors of [3] and [4] also called them modified Hermite polynomials. Here we have shown that these modified Hermite polynomials are not new polynomials, but in fact they are for Λ<0 and $\tilde{\lambda}$<0 Gegenbauer polynomials, and for Λ >0 and $\tilde{\lambda}$ >0 are pseudo-Jacobi polynomials when the second parameter of the pseudo-Jacobi polynomial is zero: $\nu = 0$. We also constructed a generating function for the pseudo-Jacobi polynomials $P_n(x; 0, N)$.

On the other hand, an analysis of the mathematical results of [10], where a quantum nonlinear harmonic oscillator was also studied, allowed us to establish a new connection between the pseudo-Jacobi and Gegenbauer polynomials. Here we have also shown that the solutions of equation (4.22) for $\lambda > 0$ can be expressed in terms of pseudo-Jacobi polynomials.

In addition, using the well-known transformation method, we have constructed two exactly solvable potentials associated with pseudo-Jacobi polynomials. These potentials were obtained earlier by other methods (see [13, 14, 18]). For them, we obtained an energy spectrum and wave functions that agree with the literature data.

APPENDIX 1

Let us prove that if in the generating function (4.10) for pseudo-Jacobi polynomials we replace x on $x/\sqrt{N}$ and go to the limit $N \to \infty$, then we obtain the generating function for the Hermite polynomials $H_n(x)$ (3.6). We will use the following approximate ($|x| \ll 1$) and asymptotic ($|z| \to \infty$, $|\arg z| < \pi$) equalities:

$$\ln(1+x) \cong x - \frac{x^2}{2}, \frac{\Gamma(z+a)}{\Gamma(z)} \cong z^a. \quad (A.1)$$

Let us first calculate the limit of the left-hand side of (4.10). We have:

$$\lim_{N\to\infty} e^{\left(N+\frac{1}{2}\right)\ln\left(1+2\sqrt{\frac{2}{N(2N+1)}}tx-\frac{2}{2N+1}t^2\right)} == \lim_{N\to\infty} e^{\left(N+\frac{1}{2}\right)\left(\frac{2}{N}tx-\frac{t^2}{N}\right)} = e^{2tx-t^2}. \quad (A.2)$$

To calculate the limit on the right-hand side of (4.10), it is sufficient to calculate the limit of the expression under the sum sign and depending on the number N. We have:

$$\lim_{N\to\infty}(-2)^n(N+1/2)^{-n}(-N-1/2)_n P_n\left(x/\sqrt{N}; 0, N\right) =$$

$$= \lim_{N\to\infty}\left(-\frac{2}{\sqrt{N}}\right)^n \frac{(-N-1/2)\Gamma(n-N+1/2)}{(n-N-1/2)\Gamma(-N+1/2)} P_n\left(x/\sqrt{N}; 0, N\right) =$$

$$= \lim_{N\to\infty}\left(-\frac{2}{\sqrt{N}}\right)^n (-N)^n P_n\left(x/\sqrt{N}; 0, N\right) = \lim_{N\to\infty}\left(2\sqrt{N}\right)^n P_n\left(x/\sqrt{N}; 0, N\right). \quad (A.3)$$

The last limit, according to equality (3.18), is equal to the Hermite polynomial $H_n(x)$. Thus, we have proved that after replacing x on $x/\sqrt{N}$ and going to the limit $N \to \infty$ in equality (3.18), it coincides with formula (3.6).

APPENDIX 2

Let us find a solution to equation (4.22) in the case $\lambda < 0$. In terms of the new variable $\xi = \sqrt{|\lambda|}x$ equation (4.22) takes the form

$$\left[(1+\xi^2)\partial_\xi^2 - 2a\xi\partial_\xi + \bar{b} + \frac{\bar{c}\xi^2}{1+\xi^2}\right]\psi = 0, \tag{A.4}$$

where we have introduced the following notations $\bar{b} = b/|\lambda|$ and $\bar{c} = c/\lambda^2$. To simplify equation (A.4), we put

$$\psi = (1+\xi^2)^{(a+1)/2}\,\phi(\xi), \tag{A.5}$$

where the function $\phi(\xi)$ obeys the following differential equation

$$\left(\partial_\xi^2 + \frac{\tilde{\tau}}{\sigma}\partial_\xi + \frac{\tilde{\sigma}}{\sigma^2}\right)\phi = 0, \tag{A.6}$$

in which $\sigma = 1+\xi^2$, $\tilde{\tau} = 2\xi$, $\tilde{\sigma} = c_0 + c_2\xi^2$. For the coefficients c_0 and c_2 we have

$$c_0 = \bar{b} + a + 1,\ \ c_2 = \bar{b} + \bar{c} - a(a+1). \tag{A.7}$$

Equation (A.6) will be solved by the Nikiforov-Uvarov method [12], that is, we are looking for its solution in the form:

$$\phi = \varphi(\xi)\,y(\xi),\ \ \varphi = e^{\int\frac{\pi(\xi)}{\sigma(\xi)}d\xi}, \tag{A.8}$$

where $\pi = \pi_0 + \pi_1\xi$ is an arbitrary polynomial of at most first degree, i.e. $\pi_{0,1} = \text{const}$. Then, one obtains the following second-order differential equation for the function $y(\xi)$:

$$y'' + \frac{\bar{\tau}}{\sigma}y' + \frac{\bar{\sigma}}{\sigma^2}y = 0, \tag{A.9}$$

with $\bar{\tau}(\xi) = \tilde{\tau}(\xi) + 2\pi(\xi)$, $\bar{\sigma}(\xi) = \tilde{\sigma} + \pi^2 + \sigma\pi'$. We choose a polynomial π from the condition that the polynomial $\bar{\sigma}(\xi)$ be divided without remainder by $\sigma(\xi)$, i.e. $\bar{\sigma} = \kappa\sigma$, $\kappa = \text{const}$. As a result we obtain the quadratic equation for the definition of a polynomial $\pi(\xi)$ and a constant κ:

$$\pi^2 - (\sigma' - \tilde{\tau})\pi - \delta\sigma + \tilde{\sigma} = 0,\ \ \delta = \kappa - \pi'. \tag{A.10}$$

From here, we find

$$\pi = e\sqrt{\delta\sigma - \tilde{\sigma}}\ ,\ \ e = \pm 1. \tag{A.11}$$

Since $\pi(\xi)$ is a polynomial, the discriminant D of a polynomial of the second degree standing under the root in (A.11) must be equal to zero. The equation $D = 0$ allows us to find a constant δ. In our case we have two solutions: 1) $\delta = c_0$, and 2) $\delta = c_2$. The physical meaning has the first solution. Thus, $\pi = e\mu\xi$, $\mu = \sqrt{c_0 - c_2}$.

After determination π, we find $\varphi(\xi)$, $\bar{\tau}(\xi)$ and κ. For $\varphi(\xi)$ we obtain the following expression: $\varphi(\xi) = (1+\xi^2)^{\frac{e\mu}{2}}$. From the requirement of finiteness $\varphi(\xi)$ at points $\xi = \pm\infty$, i.e. from the condition $\lim_{\xi\to\pm\infty}\varphi(\xi) = 0$ (square integrability condition), we get $e\mu < 0$. This means that $e = -1$ and $\mu > 0$. Thus, we obtain $\pi = -\mu\xi$ and

$$\varphi(\xi) = (1+\xi^2)^{-\frac{\mu}{2}},\ \mu = \sqrt{c_0 - c_2} = \sqrt{(a+1)^2 - \bar{c}} > 0. \tag{A.12}$$

Now, taking into account that $\bar{\tau} = 2(1-\mu)\xi$ and $\kappa = \delta + \pi' = c_0 - \mu$, one can rewrite the equation (A.9) in the form

$$(1+\xi^2)y'' + 2(1-\mu)\xi y' + (c_0 - \mu)y = 0. \tag{A.13}$$

Comparison now (A.13) with the second order differential equation (3.13) for the pseudo-Jacobi polynomials $P_n(\xi;\ \nu,\ N)$ gives us the relations

$$\nu = 0,\ N = \mu - 1,\ c_0 - \mu = n(2N + 1 - n), \tag{A.14}$$

$$y \equiv y_n(\xi) = P_n(\xi;\ 0, \mu - 1). \tag{A.15}$$

Since N is an integer and takes positive values: $N = 1,2,3, ...$, then the parameter μ is quantized, i.e. $\mu = N + 1 = 2, 3, 4, ...$ Thus, for the model (4.22) μ takes the positive integer values starting from 2. It should be borne in mind that the number n is bounded from above: $n = 0, 1, 2, 3, \dots, \mu - 1$. It follows from the last equality in (A.14) that the parameter c_0 is also quantized:

$$c_0 = \bar{b} + a + 1 = \mu(2n+1) - n(n+1). \quad \text{(A.16)}$$

If we substitute in (A.16) the values (4.23) of the parameters *a, b, c*, then for the energy spectrum of the nonlinear oscillator (4.22) we obtain the formula

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) + \frac{1}{2}|\lambda|\hbar^2[a + 1 - n(n+1)], \quad n \le \mu - 1, \quad \text{(A.17)}$$

where the renormalized oscillator frequency ω is equal to

$$\omega = \hbar|\lambda|\mu = \sqrt{k + \hbar^2\lambda^2[4\overline{\alpha\gamma} + (a+1)^2]}. \quad \text{(A.18)}$$

Formula (A.17) coincides with formula (83) in [10], however, in contrast to (83), the number of energy levels in (A.17) is finite and equal to μ.

Now, taking into account (A.5), (A.8) and (A.15) one obtains the following expression for the wave functions of the model (4.22) ($\psi \equiv \psi_{\mu n}$)

$$\psi_{\mu n}(x) = N_{\mu n}(1 + |\lambda|x^2)^{\frac{a+1}{2} - \frac{\mu}{2}} P_n\left(\sqrt{|\lambda|}x; 0, \mu - 1\right). \quad \text{(A.19)}$$

We emphasize that in the case $\lambda < 0$ the Hamiltonian of the equation (4.22) is Hermitian in the space $L^2(R, d\rho)$, where $d\rho = (1 + |\lambda|x^2)^{-a-1}\, dx$. Therefore, the wave functions (A.19) are normalized by the condition

$$\int_{-\infty}^{\infty} \psi_{\mu n}^*(x)\psi_{\mu m}(x)(1 + |\lambda|x^2)^{-a-1}\, dx = \delta_{nm}. \quad \text{(A.20)}$$

From here we find the normalization constants

$$N_{\mu n} = 2^{\mu-1-n} \frac{\Gamma(\mu-n)}{\Gamma(2\mu-1-2n)} \sqrt{\frac{\sqrt{|\lambda|}\ \Gamma(2\mu-n)}{\pi\, n!\, (2\mu-1-2n)}}. \quad \text{(A.21)}$$

Recall that in (A.21) we have $\mu - n$ =1, 2, 3, 4, … In this connection, we note that the denominator of formula (86) in [10] for the normalization constant of the wave function includes the expression $\sin^2[\pi(\mu - n)]$, which is zero. Hence, in fact, formula (86) in [10] diverges.

APPENDIX 3

Using the transformation method, we construct exactly solvable potentials in the framework of the Schrödinger equation with constant mass, which are related to the pseudo-Jacobi polynomials (3.12). The transformation method using the properties of classical orthogonal polynomials has long been used in the literature to construct exactly solvable potentials that generate solutions for bound states of the Schrödinger equation with both constant and coordinate-dependent mass [13-20]. In its simplest form, when the mass is constant, this method is as follows. Solutions of the Schrödinger equation

$$[\partial_x^2 + \frac{2m_0}{\hbar^2}(E - V)]\psi = 0 \quad \text{(A.22)}$$

are associated with orthogonal polynomials $F(g(x))$ as

$$\psi(x) = f(x)\, F(g(x)). \quad \text{(A.23)}$$

In general, $F(g)$ can be any special function satisfying the second order differential equation

$$[\partial_g^2 + Q(g)\partial_g + R(g)]F(g) = 0\ . \quad \text{(A.24)}$$

As a result, we obtain the following relation between the coefficients of equations (A.22) and (A.24) (see for more details, for example, [13])

$$\frac{2m_0}{\hbar^2}(E - V(x)) = A_0 + (g')^2\left[R(g) - \frac{1}{2}\partial_g Q(g) - \frac{1}{4}Q^2(g)\right],$$

$$A_0 = \frac{g'''}{2g'} - \frac{3}{4}\left(\frac{g''}{g'}\right)^2, \quad f(x) = (g')^{-1/2}\exp\left(\frac{1}{2}\int Q(g)g'dx\right). \quad \text{(A.25)}$$

For pseudo-Jacobi polynomials we have (see (3.13))

$$Q(g) = \frac{2(\nu - Ng)}{1+g^2}, \quad R(g) = \frac{n(2N+1-n)}{1+g^2}. \quad \text{(A.26)}$$

Substitution of (A.26) in (A.25) leads to the expression

$$\frac{2m_0}{\hbar^2}\big(E - V(x)\big) = A_0 - a_1 A_1 + a_2 A_2 + a_3 A_3, \tag{A.27}$$

where the coefficients a_i and the quantities A_i $(i = 1, 2, 3)$ are equal to

$$a_1 = n(n - 2N - 1) + N(N + 1), \tag{A.28}$$

$$a_2 = N(N + 2) - \nu^2, \quad a_3 = 2\nu(N + 1), \tag{A.29}$$

$$A_1 = \frac{(g')^2}{1+g^2}, \quad A_2 = \frac{(g')^2}{(1+g^2)^2}, \quad A_3 = \frac{(g')^2 g}{(1+g^2)^2}. \tag{A.30}$$

In order for equality (A.27) to hold for all values of x, one of the terms containing the function $g(x)$ or its derivative is assumed to be constant. This constant determines the value of the energy of the considered quantum system. This procedure is an integral part of this method. A constant term can therefore be generated on the right-hand side of eq. (A.27) by assuming 1) $A_1 = a^2$, 2) $A_2 = a^2$ and 3) $A_3 = a^2$, where $a^2 > 0$.

We will consider only the first two cases. For the reason indicated in [13], the third case does not lead to a new potential.

1) In the first case, for the function $g(x)$ the equation $A_1 = a^2$ gives the following value $g(x) = \sinh(ax + b)$, were b is a constant of integration. In this case, the energy $E \equiv E_n$and potential $V(x)$ take the form

$$E_n = \frac{\hbar^2 a^2}{2m_0}[n(2N + 1 - n) - N(N + 1) - 1/4], \; n = 0, 1, 2, 3, \dots, N, \tag{A.31}$$

$$V(x) = \frac{\hbar^2 a^2}{2m_0}[(\nu^2 - N(N + 2) - 3/4)\text{sech}^2(ax + b) - \\ -2\nu(N + 1)\,\text{sech}(ax + b)\tanh(ax + b)]. \tag{A.32}$$

The wave functions corresponding to the energy levels (A.31) are

$$\psi_n = C_n(1 + g^2)^{-\frac{N}{2}-\frac{1}{4}}\, e^{\nu \arctan(g)} P_n(g; \nu, N). \tag{A.33}$$

where $g = \sinh(ax + b)$. They are normalized by the condition (expression for $d_{Nn}(\nu)$ given in (3.14))

$$\int_{-\infty}^{\infty} \psi_n^*(x)\psi_m(x)dx = \delta_{nm}, \quad C_n = \frac{\sqrt{a}}{d_{Nn}(\nu)}. \tag{A.34}$$

If in (A.31) and (A.32) we introduce new notaion $A = sa$ and $B = \lambda a$, where $s = N + 1/2$, $\lambda = -\nu$, and also put $b = 0, 2m_0 = \hbar = 1$, we get

$$E_n = -(A - na)^2, \quad n = 0, 1, 2, 3, \dots, s - 1/2, \tag{A.35}$$

$$V(x) = (B^2 - A^2 - Aa)\text{sech}^2(ax) + B(2A + a)\,\text{sech}(ax)\tanh(ax). \tag{A.36}$$

These formulas were obtained in [13, 14, 18]. Here we will only note the following: 1) the parameter s (hence the potential parameter A) takes discrete values equal to $= \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \dots$; 2) the number of energy levels of the system is finite and equal to $s + 1/2$.

2) In the second case, for the function $g(x)$ from the equation $A_2 = a^2$ we obtain $g(x) = \tan(ax + b)$. In this case, the constant term in (A.27) does not depend on the number *n*, i.e.

$$\frac{2m_0}{\hbar^2}E = a^2[N(N + 2) - \nu^2 + 1], \tag{A.37}$$

$$\frac{2m_0}{\hbar^2}V(x) = -a^2[n(n - 2N - 1) + N(N + 1)]\sec^2(ax + b) + \\ +2a^2\nu(N + 1)\tan(ax + b). \tag{A.38}$$

Introducing $s = N + 1 - n$ and $\lambda = \nu(s + n)$ as new parameters we can transfer the n dependence to the constant term E. Then we obtain we obtain the well-known formulas [13, 14, 18]:

$$E_n = \frac{\hbar^2}{2m_0}\left[(A + na)^2 - \frac{B^2}{(A+na)^2}\right], \quad n = 0, 1, 2, 3, \dots, \tag{A,39}$$

$$V(x) = \frac{\hbar^2}{2m_0}[A(A - a)\sec^2(ax + b) - 2B\tan(ax + b)], \tag{A.40}$$

where $A = sa$ and $B = \lambda a^2$.

Let us now write out the wave functions corresponding to the energy levels (A.39). They look like

$$\psi_n = C_n(1+g^2)^{-\frac{s+n}{2}} \; e^{\frac{\lambda}{s+n}(ax+b)} \, P_n\left(g; \frac{\lambda}{s+n}, s+n-1\right), \tag{A.41}$$

where $g = \tan(ax+b)$ and $-\frac{\pi}{2} < ax+b < \frac{\pi}{2}$. Note that the condition $n \le N$ or $n \le s+n-1$, which is valid for the pseudo-Jacobi polynomials (A.41), holds for all nonnegative integer values of the number $n = 0, 1, 2, 3, \ldots$,if $s \ge 1$. Therefore, the wave functions (A.41) are square integrable for any nonnegative integer values of n. However, the question of proving the orthogonality ψ_n and ψ_m requires additional analysis (see formula (3.27) in [18]).

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