THE WIGNER DISTRIBUTION FUNCTION OF A SEMICONFINED HARMONIC OSCILLATOR MODEL WITH A POSITION-DEPENDENT MASS AND FREQUENCY IN AN EXTERNAL HOMOGENEOUS FIELD. THE CASE OF PARABOLIC WELL

SHAKIR M. NAGIYEV, SHOVQIYYA A. AMIROVA

Institute of Physics Ministry of Science and Education of Azerbaijan, AZ 1143, Baku, H. Javid Ave., 131

The phase space representation for a semiconfined harmonic oscillator model with the position-dependent mass and frequency in an external homogeneous field is constructed in terms of the Wigner distribution function. It is expressed through the Bessel function and Laguerre polynomials. Some of the special cases and limits are also discussed.

Keywords: Harmonic oscillator; external homogeneous field; position-dependent mass; Wigner function; limit relations.
PACS: 03.65.-w; 02.30. Hq; 03.65.Ge

1. INTRODUCTION

As is known, one of the formulations of quantum mechanics is its formulation in the phase space [1-5]. This formulation uses concepts that are common to both quantum and classical mechanics. It makes it possible to describe the picture of quantum phenomena using, as far as possible, the classical language. This formulation deals only with c-numerical quantities and equations, and not with operators, which sometimes simplifies the mathematical description of a given quantum system. The main tools of the phase formulation of quantum mechanics are quantum distribution functions. To pass to this formulation from the Schrödinger operator formalism, it is necessary to replace the operators of physical quantities with their Weyl transformations [6], and the wave functions with quantum distribution functions.

Among the various quantum distributions functions that exist, the Wigner quantum distribution function is well known. Wigner function W depends on momentum p and coordinates x particles and in the general case on time t, those $W=W(p,x,t)$. Introduced in 1932, the Wigner function is widely used to describe various physical quantum systems. The Wigner quantum distribution function is expressed in terms of the Schrödinger wave function $\psi(x,t)$ using the formula:

$$W(x,p,t)=\frac{1}{2\pi\hbar}\int\psi^{*}(x-\frac{x'}{2},t)\psi(x+\frac{x'}{2},t)e^{-ipx'/\hbar}\,dx. \quad (1.1)$$

At present there exist a lot of papers with computation of the Wigner function of the various constant [7-10] and position-dependent mass [11-17] quantum relativistic and nonrelativistic harmonic oscillator models.

On the other hand, it is also well known that the construction and study of models of dynamic quantum physical systems with coordinate-dependent mass has long attracted the attention of scientists [18–37]. Such quantum systems play an important role in studying the physical and electronic properties of semiconductors [22], quantum wells and quantum dots [23], clusters ^{3}He [24], quantum liquids [25], graded alloys, semiconductor heterostructures [26], etc.

The aim of this work is to construct the Wigner quantum distribution function for a linear oscillator model with a position-dependent mass and frequency in an external homogeneous field [38].

2. WIGNER QUANTUM DISTRIBUTION FUNCTION FOR A LINEAR HARMONIC OSCILLATOR WITH CONSTANT MASS IN A UNIFORM EXTERNAL FIELD

Let us write the Schrödinger equation describing the motion of a linear harmonic oscillator with a constant mass in a uniform external field

$$\left(\frac{\hat{p}^2}{2m_0}+\frac{m_0\omega_0^2x^2}{2}+gx\right)\psi^{HO}(x)=E^{\mathrm{HO}}\psi^{\mathrm{HO}}(x), \quad (2.1)$$

where $\hat{p}=-i\hbar\partial_x$ is the momentum operator, m_0 and ω_0 are constant mass and frequency of the oscillator, equation (2.1) is defined on the entire real axis $-\infty<x<\infty$.

It is well known that the exact solution of Eq. (2.1) is expressed in terms of the Hermite polynomials

$$\psi_n^{\mathrm{HO}}(x)=c_n^{\mathrm{HO}}e^{-\frac{1}{2}(\xi+\xi_0)^2}H_n(\xi+\xi_0), n=0,1,2,3\ ..., \quad (2.2)$$

and the discrete energy spectrum corresponding to the wave functions is equidistant

131, H.Javid ave, AZ-1143, Baku
Institute of Physics
E-mail: jophphysics@gmail.com

$$E_n^{\text{HO}} = \hbar\omega_0\left(n+\frac{1}{2}\right)+\frac{m_0\omega_0^2x^2}{2},\ n = 0,1,2,3\ ... \tag{2.3}$$

Here we use the following notation

$$\xi = \lambda_0 x, \xi_0 = \lambda_0 x_0, x_0 = \frac{g}{m_0\omega_0^2}, \lambda_0 = \sqrt{\frac{m_0\omega_0}{\hbar}}. \tag{2.4}$$

From the orthonormalization condition for wave functions (2.2)

$$\int_{-\infty}^{\infty}\psi_n^{*\text{HO}}(x)\,\psi_m^{\text{HO}}(x)dx = \delta_{nm} \tag{2.5}$$

we find the normalization constant as follows

$$c_n^{\text{HO}} = \sqrt[4]{\frac{\lambda_0^2}{\pi}}\frac{1}{\sqrt{2^n n!}}. \tag{2.6}$$

Substituting (2.2) into (1.1) leads to the following expression for the Wigner function for a linear harmonic oscillator in an external uniform field

$$W_n^{\text{HO}}(p,x) = \frac{(-1)^n}{\pi\hbar}e^{-(\xi+\xi_0)^2-\eta^2}L_n(2\eta^2+2(\xi+\xi_0)^2), \tag{2.7}$$

where $L_n(x)$ are Laguerre polynomials, and $\eta = p/\lambda_0\hbar$. Formula (2.7) can also be written in operator form [11].

$$W_n^{\text{HO}}(p,x) = \frac{1}{\pi\hbar}\frac{1}{2^n n!}H_n\left(\xi+\xi_0-\frac{i}{2}\partial_\eta\right)H_n\left(\xi+\xi_0+\frac{i}{2}\partial_\eta\right)\ e^{-(\xi+\xi_0)^2-\eta^2}. \tag{2.8}$$

2. THE LINEAR HARMONIC OSCILLATOR WITH POSITION DEPENDENT MASS AND FREQUENCY IN THE EXTERNAL HOMOGENEOUS FIELD WITH A LIMITED PARABOLIC WELL

Linear harmonic oscillator model with a position dependent mass $M(x) = \frac{a^2m_0}{(a+x)^2}$ and frequency $\omega = \omega_0\left(1+\frac{x}{a}\right)$, $(a+x>0)$ in an external uniform field $V_{ext}(x) = g(x)$, considered in [38] is described by the Schrödinger equation

$$\left[H_0+V_{eff}(x)\right]\psi(x) = E\psi(x), \qquad a+x>0. \tag{3.1}$$

Here H_0 is the free Hamiltonian with a position dependent mass

$$H_0 = \frac{1}{2}\hat{p}\frac{1}{M(x)}\hat{p}+V_{free}(x), \tag{3.2}$$

and $V_{free}(x)$ is the contribution from the free Hamiltonian to the potential energy, which depends on the mass function $M(x)$ and on the real parameters $A_f, B_f \in R(-\infty,\infty)$(see [38,39,40]). It has a form

$$V_{free}(x) = A_f\frac{\hbar^2 {M'}^2}{2M^3}-B_f\frac{\hbar^2M''}{4M^2}. \tag{3.3}$$

The effective potential is equal to the sum of the interaction potential $V(x)$ and free potential, i.e.

$$V_{eff}(x) = V(x)+V_{free}(x). \tag{3.4}$$

Interaction potential $V(x)$ we choose in the form

$$V(x) = \begin{cases}\frac{M(x)\omega^2x^2}{2}+gx, & x+a>0,\\ \infty, & x+a<0,\end{cases} \tag{3.5}$$

where g is a force and $M(x)\omega^2(x) = m_0\omega_0^2$.

In this case, when $M(x) = \frac{a^2m_0}{(a+x)^2}$ we have $V_{free}(x) = V_0 = \text{const}$. The solution of the equation (3.1) with the potential (3.4)

$$\psi_n^g(x) = c_n^g \left(1+\frac{x}{a}\right)^{A_n} e^{-b^2\left(1+\frac{x}{a}\right)} L_n^{2A_n+1}\left(2b^2\left(1+\frac{x}{a}\right)\right), n = 0,1,2,\dots N_g, \quad (3.6)$$

is expressed in terms of the Laguerre polynomials [38]

$$L_n^\alpha(x) = \frac{(\alpha+1)_n}{n!}\ {}_1F_1(-n, \alpha+1; x)\ , \quad (3.7)$$

in the following way, where $b = \lambda_0 a$, $A_n = b^2 - n - 1 - b\xi_0$, $N_g = b^2 - 1 - b\xi_0$. The corresponding to the wave functions (3.7) discrete energy spectrum has the form

$$E_n^g = \hbar\Omega_0\left(n+\frac{1}{2}\right) - \frac{\hbar^2}{2m_0a^2}n(n+1) - \frac{m_0\omega_0^2x_0^2}{2} + V_0, \quad (3.8)$$

where $\Omega_0 = \omega_0\left(1-\frac{\xi_0}{b}\right)$ is a renormalized oscillator frequency.

Let us now find the normalization constant c_n^g in (3.7) from the condition

$$\int_{-a}^{\infty}\left|\psi_n^g(x)\right|^2 dx = 1. \quad (3.9)$$

To calculate this integral (3.9), we use the formula [41]

$$\int_0^\infty x^{\alpha-1}e^{-cx}L_m^\gamma(cx)L_n^\lambda(cx)dx = \frac{(\gamma+1)_n(\lambda-\alpha+1)_n}{m!n!c^\alpha}\Gamma(\alpha)\ {}_3F_2(-m,\alpha,\alpha-\lambda;\gamma+1;\alpha-\lambda-n;1), \quad (3.10)$$

$\mathrm{Re}\alpha > 0$, $\mathrm{Re}c > 0$. In our case we have $\alpha = \gamma = \lambda = 2A_n + 1$, $m = n, c = 1$. As a result, we find

$$C_n^g = \left(2b^2\right)^{A_n+\frac{1}{2}}\sqrt{\frac{\lambda_0 n!(2A_n+1)}{b\Gamma(2A_n+n+1)}}\ . \quad (3.11)$$

4. COMPUTATION OF THE WIGNER DISTRIBUTION FUNCTION OF A SEMICONFINED LINEAR HARMONIC OSCILLATOR MODEL WITH POSITION-DEPENDENT MASS AND FREQUENCY IN AN EXTERNAL HOMOGENEOUS FIELD

For our calculations, we will substitute expression (3.7) for the wave function into the definition of the Wigner distribution function

$$W_n^g(p,x) = \frac{\left|C_n^g\right|^2}{\pi\hbar}e^{-2b^2\left(1+\frac{x}{a}\right)}\int_{y_1}^{y_2}\left[\left(1+\frac{x}{a}\right)^2 - \frac{y^2}{a^2}\right]^{A_n} L_n^{2A_n+1}\left(2b^2\left(1+\frac{x-y}{a}\right)\right)\times$$
$$\times L_n^{2A_n+1}\left(2b^2\left(1+\frac{x+y}{a}\right)\right)e^{-\frac{2ipy}{\hbar}}dy. \quad (4.1)$$

Integration limits in (4.1) y_1 and y_2 we find from the condition that the argument of the wave function varies in the region $x > -a$, therefore, we have $x - y > -a$ and $x + y > -a$. Hence it follows that $y_1 = -(x+a), y_2 = (x+a)$. Since the boundaries of integration in (4.1) are finite, this integral converges.

We introduce a dimensionless variable $t = y/(x+a)$ and imagine $W_n^g(p,x)$ as

$$W_n^g(p,x) = \sigma_n \cdot I_n, \quad (4.2)$$

$$\sigma_n = \frac{|C_n^g|^2}{\pi\hbar\lambda_0}e^{-\rho}b^{-2A_n}(b+\xi)^{2A_n+1} = \frac{1}{\pi\hbar}e^{-\rho}[2b(b+\xi)]^{2A_n+1}\cdot\frac{n!\,(2A_n+1)}{\Gamma(2A_n+n+2)}.$$

$$I_n = \int_{-1}^{1}(1-t^2)^{A_n}\,L_n^{2A_n+1}(\rho-\rho t)L_n^{2A_n+1}(\rho+\rho t)e^{-2i\eta(b+\xi)t}dt,$$

where $\rho = 2b(b+\xi), \eta = p/\lambda_0\hbar$. Using equality

$$te^{-2i\eta(b+\xi)t} = \frac{i}{2(b+\xi)}\partial_\eta e^{-2i\eta(b+\xi)t}$$

rewrite I_n in operator form

$$I_n = L_n^{2A_n+1}\left(\rho - ib\partial_\eta\right) L_n^{2A_n+1}\left(\rho + ib\partial_\eta\right) \int_{-1}^{1} (1-t^2)^{A_n}\, e^{-2i\eta(b+\xi)t} dt. \quad (4.3)$$

Here integration can be carried out using the formulas [41]

$$\int_{-a}^{a} (a^2 - x^2)^{\beta-1} e^{i\lambda x} dx = \sqrt{\pi}\Gamma(\beta)\left(\frac{2a}{\lambda}\right)^{\beta-1/2} J_{\beta-1/2}(a\lambda), Re\beta > 0, \quad (4.4)$$

where $J_\beta(x)$ is the Bessel function. As a result, we obtain the following operator relation

$$I_n = \sqrt{\pi}\Gamma(A_n+1) L_n^{2A_n+1}\left(\rho - ib\partial_\eta\right) L_n^{2A_n+1}\left(\rho + ib\partial_\eta\right) [\eta(b+\xi)]^{-A_n-1/2} J_{A_n+1/2}\left(2\eta(b+\xi)\right). \quad (4.5)$$

Taking into account (4.5), we can now write the Wigner function in operator form, i.e.

$$W_n^g(p,x) = \frac{|C_n^g|^2}{\hbar\sqrt{\pi}\lambda_0}\Gamma(A_n+1) b^{-2A_n} e^{-\rho} L_n^{2A_n+1}\left(\rho - ib\partial_\eta\right) L_n^{2A_n+1}\left(\rho + ib\partial_\eta\right) \times$$
$$\times \left(\frac{b+\xi}{\eta}\right)^{A_n+1/2} J_{A_n+1/2}\left(2\eta(b+\xi)\right). \quad (4.6)$$

4.1. GROUND STATE WIGNER FUNCTION

From (3.16) we extract the Wigner function of the ground state, which has the form

$$W_0^g(p,x) = \frac{1}{\hbar\sqrt{\pi}}(2A_0+1)\frac{\Gamma(2A_0+1)}{\Gamma(2A_0+2)}\left(\frac{4b^2(b+\xi)}{\eta}\right)^{A_0+1/2} e^{-\rho} J_{A_0+1/2}\left(2\eta(b+\xi)\right). \quad (4.7)$$

Now taking into account the following well-known relation for the Gamma functions [43]

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \quad (4.8)$$

we can simplify expression (4.7). As a result we obtain the following analytical expression for the ground state Wigner distribution function

$$W_0^g(p,x) = \frac{2}{\hbar}\frac{1}{\Gamma(2A_0+1/2)}\left(\frac{b^2(b+\xi)}{\eta}\right)^{A_0+1/2} e^{-\rho} J_{A_0+1/2}\left(2\eta(b+\xi)\right) \quad (4.9)$$

or explicitly

$$W_0^g(p,x) = \frac{2}{\hbar}\frac{1}{\Gamma(b^2-b\xi_0-1/2)} e^{-b^2-b\xi_0}\left(\frac{b^2(b+\xi)}{\eta}\right)^{b^2-b\xi_0-1/2} J_{b^2-b\xi_0-1/2}\left(2\eta(b+\xi)\right), \quad (4.10)$$

where $A_n = A_0 - n$, $A_0 = b^2 - b\xi_0 - 1$.

For the case of the absence of the external field $g = 0$ and $\xi_0 = 0$, and the Wigner function of the ground state (4.8) slightly simplifies as follows:

$$W_0^0(p,x) = \frac{2}{\hbar}\frac{1}{\Gamma(b^2-1/2)} e^{-b^2-b\xi_0}\left(\frac{b^2(b+\xi)}{\eta}\right)^{b^2-1/2} J_{b^2-1/2}\left(2\eta(b+\xi)\right). \quad (4.11)$$

Taking into account that the Wigner function of the ground state (4.2) is exactly computed in terms of the Bessel functions, then one can try to compute its analytical expression for arbitrarily excited states n. For this, one needs to go to the expression (4.2). Its integrand mainly consists of the product of two Lagerre polynomials with different arguments. One applies there the following known finite sum for such kind of products [42]

$$L_n^\alpha(x) L_n^\alpha(y) = \frac{\Gamma(\alpha+n+1)}{n!}\sum_{k=0}^{n}\frac{(xy)^k}{k!\Gamma(\alpha+k+1)} L_{n-k}^{(\alpha+2k)}(x+y). \quad (4.12)$$

Its substitution at (4.2) yields

$$I_n = \frac{\Gamma(2A_n+n+2)}{n!}\sum_{k=0}^{n}\frac{\rho^{2k}}{k!\Gamma(2A_n+n+2)}L_{n-k}^{(2A_n+n+2)}(2\rho)\int_{-1}^{1}(1-t^2)^{A_n+k}\, e^{-2i\eta(b+\xi)t}dt. \quad (4.13)$$

Applying again the integral formula (4.4), for I_n we get

$$I_n = \sqrt{\pi}\frac{\Gamma(2A_n+n+2)}{n!}[\eta(b+\xi)]^{A_n-1/2}\sum_{k=0}^{n}Q_k, \quad (4.14)$$

where

$$Q_k = \frac{\Gamma(A_n+k+1)}{k!\Gamma(2A_n+k+2)}\left[\frac{4b^2(b+\xi)}{\eta}\right]^k L_{n-k}^{(2A_n+2k+1)}(2\rho)J_{A_n+k+1/2}\big(2\eta(b+\xi)\big). \quad (4.15)$$

So the Wigner distribution function of the semiconfined quantum harmonic oscillator model with position dependent mass and frequency in the presence of the external homogeneous field takes a form

$$W_n^g(p,x) = \frac{1}{\hbar\sqrt{\pi}}(A_n+1)e^{-\rho}\times$$

$$\times\sum_{k=0}^{n}\frac{\Gamma(A_n+k+1)}{k!\Gamma(2A_n+k+2)}\left[\frac{2b\rho}{\eta}\right]^{A_n+k+1/2} L_{n-k}^{(2A_n+2k+1)}(2\rho)J_{A_n+k+1/2}\big(2\eta(b+\xi)\big), \quad (4.16)$$

or explicitly

$$W_n^g(p,x) = \frac{1}{\hbar\sqrt{\pi}}(2b^2-2b\xi_0-2n-1)e^{-2b(b+\xi)}\sum_{k=0}^{n}\frac{\Gamma(b^2-b\xi_0-n+k)}{k!\,\Gamma(2b^2-2b\xi_0-2n+k)}\times$$

$$\times\left[\frac{4b^2(b+\xi)}{\eta}\right]^{b^2-b\xi_0-n+k-1/2} L_{n-k}^{2b^2-2b\xi_0-2n+2k-1}\big(2b(b+\xi)\big)J_{b^2-b\xi_0-n+k-1/2}\big(2\eta(b+)\big). \quad (4.17)$$

Absence of the external field again slightly simplifies (4.17) due to that $g = 0$ ($\xi_0 = 0$):

$$W_n^0(p,x) = \frac{1}{\hbar\sqrt{\pi}}(2b^2-2n-1)e^{-2b(b+\xi)}\sum_{k=0}^{n}\frac{\Gamma(b^2-n+k)}{k!\,\Gamma(2b^2-2n+k)}\times$$

$$\times\left[\frac{4b^2(b+\xi)}{\eta}\right]^{b^2-n+k-1/2} L_{n-k}^{2b^2-2n+2k-1}\big(2b(b+\xi)\big)J_{b^2-n+k-1/2}\big(2\eta(b+\xi)\big). \quad (4.18)$$

We obtained an exact expression for the Wigner distribution function of our model of the linear harmonic oscillator with position dependent mass and frequency in the external homogeneous field.

5. LIMIT CASE $a \to \infty$ (or $b \to \infty$)

In this section, we will find the limit of the wave function (3.6) and the Wigner distribution function (4.6). In doing so, we will proceed from asymptotic formulas valid for $|x| \ll 1$ and $|z| \to \infty$:

$$\sqrt{1+x} = 1+\tfrac{1}{2}x-\tfrac{1}{8}x^2,\ \ln(1\pm x) \cong \pm x-\tfrac{1}{2}x^2,$$
$$\Gamma(z+1) \cong \sqrt{2\pi z}\, e^{z\ln z - z}, \quad (5.1)$$

as well as the limit formula for Laguerre polynomials [44]

$$\left(\frac{\alpha}{2}\right)^{\frac{n}{2}} L_n^{\alpha}\big(\alpha+\sqrt{2\alpha}x\big) = \frac{(-1)^n}{n!}H_n(x). \quad (5.2)$$

5.1. Limit of wave functions. To calculate the limit of the wave function (3.6) at $a \to \infty$, first, we find the asymptotics of each of the factors separately in (3.6).

a) In c_n^g for the Gamma function we have

$$\Gamma(2A_n + k + 2) = \Gamma(2b^2 - 2b\xi_0 - n) \cong \frac{\sqrt{\pi}}{b} e^{\sigma_1},$$
$$\sigma_1 \cong (2b^2 - 2b\xi_0 - n)\ln 2b^2 + \xi_0^2 - 2b^2.$$

Hence,

$$C_n^g \cong \frac{\sqrt[4]{\lambda_0^2}}{\pi}\sqrt{n!}\,(2b^2)^{-n/2} e^{-b^2-\frac{1}{2}\xi_0^2}. \quad (5.3)$$

$$\text{b)}\left(1+\frac{\xi}{b}\right)^{A_n} = e^{A_n \ln\left(1+\frac{\xi}{b}\right)} \cong e^{\sigma_2}, \sigma_2 \cong b\xi - \frac{1}{2}\xi^2 - \xi\xi_0. \quad (5.4)$$

Substituting (5.3) and (5.4) into (3.6), we have for $b \to \infty$

$$\psi_n^g(x) \cong \frac{\sqrt[4]{\lambda_0^2}}{\pi}\sqrt{n!}\, e^{-\frac{1}{2}(\xi+\xi_0)^2}(2b^2)^{-n/2} L_n^{2b^2-2b\xi_0-2n-1}(2b^2+2b\xi). \quad (5.5)$$

Now to calculate the limit

$$Z_n = \lim_{b\to\infty}(2b^2)^{-n/2} L_n^{2b^2-2b\xi_0-2n-1}\left(2b^2+2b\xi\right), \quad (5.6)$$

we introduce the notation $P_n = 2b^2 - 2b\xi_0 - 2n - 1, \alpha = 2b^2, z = 2b^2 + 2b\xi$ and obtain the recurrent formula for the Laguerre polynomials $L_n^{P_n}(z)$.

Since for for the Laguerre polynomials $L_n^{\alpha}(z)$ the recurrence relation is valid [42]

$$(n+1)L_{n+1}^{\alpha}(z) - (2n+\alpha+1-z)L_n^{\alpha}(z) + (n+\alpha)L_{n-1}^{\alpha}(z) = 0, \quad (5.7)$$

then, the recurrence relation for $L_n^{P_n}(z)$ will have

$$(n+1)L_{n+1}^{P_{n+1}}(z) - (2n+P_n+1-z)L_n^{P_{n+1}}(z) + (n+P_{n+1})L_{n-1}^{P_{n+1}}(z) = 0, \quad (5.8)$$

We now prove by mathematical induction the following limit relation

$$Z_n = \lim_{\alpha\to\infty}(\alpha)^{-n/2} L_n^{\alpha-\sqrt{2\alpha}\xi_0-2n-1}\left(\alpha+\sqrt{2\alpha}\xi_0\right) = \frac{(-1)^n}{n!}H_n(\xi+\xi_0). \quad (5.9)$$

Proof. First we write explicitly the Laguerre and Hermite polynomials for the first few values n

$$L_0^{\alpha}(z) = 1, L_1^{\alpha}(z) = 1+\alpha-z, L_2^{\alpha}(z) = \frac{1}{2}[(1+\alpha-z)(3+\alpha-z)-1-\alpha],$$
$$H_0(z) = 1, H_1(z) = 2z, H_2(z) = 4z^2 - 2. \quad (5.10)$$

We will also need a recurrence relation for the Hermite polynomial

$$H_{n+1}(z) = 2zH_n(z) - 2nH_{n-1}(z). \quad (5.11)$$

Using these expressions, we directly obtain that for $n = 1$ and $n = 2$ relation (5.9) is true:

$$Z_1 = \lim_{\alpha\to\infty}(\alpha)^{-1/2} L_1^{\alpha-\sqrt{2\alpha}\xi_0-3}\left(\alpha+\sqrt{2\alpha}\xi_0\right) = -\sqrt{2}(\xi+\xi_0) = -\frac{1}{\sqrt{2}}H_1(\xi+\xi_0)$$
$$Z_2 = \lim_{\alpha\to\infty}(\alpha)^{-1} L_2^{\alpha-\sqrt{2\alpha}\xi_0-5}\left(\alpha+\sqrt{2\alpha}\xi_0\right) = (\xi+\xi_0)^2 - \frac{1}{2} = \frac{1}{4}H_2(\xi+\xi_0). \quad (5.12)$$

Let us now prove that relation (5.9), which is valid in the cases $n = 1$ и $n = 2$, also performed for an arbitrary $n > 2$. For this, we assume that relation (5.9) holds for the polynomials $L_{n+1}^{P_{n+1}}(z)$ and $L_{n-1}^{P_{n+1}}(z)$ at some n. Then it also holds for $L_{n+1}^{P_{n+1}}(z)$. Indeed, we multiply by $\alpha^{-\frac{n+1}{2}}$ both sides of the recurrence relation for Laguerre polynomials $L_{n+1}^{P_{n+1}}(z)$ (5.8)

$$Z_{n+1} = \lim_{\alpha\to\infty}\alpha^{-\frac{n+1}{2}} L_1^{\alpha-\sqrt{2\alpha}\xi_0-2n-3}\left(\alpha+\sqrt{2\alpha}\xi_0\right) =$$

$$= \frac{(-1)^{n+1}}{(n+1)!\sqrt{2^{n+1}}}[2(\xi+\xi_0)H_n(\xi+\xi_0) - 2nH_{n-1}(\xi+\xi_0)]. \tag{5.13}$$

According to (5.11), the last expression is

$$Z_{n+1} = \frac{(-1)^{n+1}}{(n+1)!\sqrt{2^{n+1}}}H_{n+1}(\xi+\xi_0). \tag{5.14}$$

This completes the proof.

5.2. Limit of the Wigner distribution function for the ground state.

To calculate this limit, we find the asymptotics of each of the factors in (4.10). We have

$$\text{a) } \frac{1}{\Gamma\left(b^2-b\xi-1/2\right)} \cong \frac{1}{\sqrt{2\pi}}\left(b^2\right)^{-A_0} e^{b^2-\frac{1}{2}\xi_0^2}, \tag{5.15a}$$

$$\text{b) } \left[\frac{b^2(b+\xi)}{\eta}\right]^{b^2-b\xi_0^2-1/2} \cong \left(\frac{b^3}{\eta}\right)^{A_0+1/2} e^{b\xi-\frac{1}{2}\xi^2-\xi\xi_0}, \tag{5.15b}$$

$$\text{c) } J_{b^2-b\xi_0-1/2}\left(2\eta(b+\xi)\right) \cong \frac{1}{b\sqrt{2\pi}}\left(\frac{\eta}{b}\right)^{A_0+1/2} e^{b^2+b\xi-\eta^2-\xi_0^2-\xi\xi_0}. \tag{5.15c}$$

Note that the asymptotic behavior of the Bessel function in (5.15c) was found using the asymptotic formula (7.13, 8(14)) for $J_p(x)$, given in [42]

$$J_p(x) \cong \frac{1}{\sqrt{2\pi}\sqrt[4]{p^2-x^2}}\exp\left(\sqrt{p^2-x^2} - p\text{Arch}\frac{p}{x}\right), p > x > 0, p \to \infty. \tag{5.16}$$

We emphasize that there is a typo in the formula (7.13,8(14)) in [42]: instead of Arsh$\frac{p}{x}$ should stand Arch$\frac{p}{x}$. We took this correction into account in formula (4.16). To obtain (5.15c) we left in (5.16) the main terms in powers b^{-1}:

$$\sqrt[4]{p^2-x^2} \cong b, \quad \sqrt{p^2-x^2} \cong A_0 + \frac{1}{2} - 2\eta^2,$$
$$\text{Arch}\frac{p}{x} \cong \ln\left(\frac{b}{\eta}\right) - \frac{\xi+\xi_0}{b} + \frac{\xi^2-1-2\eta^2-\xi_0^2}{2b^2}.$$

As a result of substituting asymptotics (5.15) into (3.21), we find that the limit of the Wigner distribution function of the ground state coincides with formula (2.7) for $n = 0$, i.e.

$$\lim_{b\to\infty} W_0^g(p,x) = \frac{1}{\pi\hbar}e^{-\eta^2-(\xi+\xi_0)^2} = W_0^{\text{HO}}(p,x). \tag{5.17}$$

5.3. Limit of the Wigner distribution function for n excited state. To calculate the limit of $W_n^g(p,x)$ it is convenient to start from equality (4.6). Let's rewrite it in the form

$$W_n^g(p,x) = L_n^{2A_n+1}\left(\rho - ib\partial_\eta\right)L_n^{2A_n+1}\left(\rho + ib\partial_\eta\right)\Omega_n^g(p,x), \tag{5.18}$$

where

$$\Omega_n^g(p,x) = \frac{1}{\hbar\sqrt{\pi}}\frac{n!(2A_n+1)\Gamma(A_n+1)}{\Gamma(2A_n+n+2)}e^{-\rho}\left(\frac{4b^3}{\eta}\right)^{A_n+1/2}\left(1+\frac{\xi}{b}\right)^{A_n+1/2}J_{A_n+1/2}\left(2\eta(b+\xi)\right). \tag{5.19}$$

As above, we find the asymptotics of each factor (5.19) as $b \to \infty$. We have

$$\text{a) } \Gamma(A_n+1) \cong \sqrt{2\pi}(b^2)^{A_n+1/2}e^{\gamma_1}, \gamma_1 = -b^2 + \frac{\xi_0^2}{2},$$

$$\text{b) } \frac{(2A_n+1)}{\Gamma(2A_n+n+2)} \cong \frac{1}{\Gamma(2A_n+n+1)} \cong \frac{1}{2b\sqrt{\pi}}(2b^2)^{-2A_n-n}e^{\gamma_2}, \gamma_2 = 2b^2 - \xi_0^2, \tag{5.20}$$

$$\text{c) } \left(1+\frac{\xi}{b}\right)^{A_n+1/2} \cong e^{\gamma_3}, \gamma_3 = b\xi - \frac{\xi^2}{2} - \xi\xi_0,$$

d) $J_{A_n+1/2}\big(2\eta(b+\xi)\big) \cong \frac{1}{b\sqrt{2\pi}}\left(\frac{\eta}{b}\right)^{A_0+1/2} e^{\gamma_4}, \gamma_4 = b^2 + b\xi - \eta^2 - \frac{1}{2}(\xi+\xi_0)^2.$

So for $\Omega_n^g(p,x)$ we get the following asymptotics

$$\Omega_n^g(p,x) \cong \frac{1}{\pi\hbar} n!\,(2b^2)^{-n} e^{-\eta^2-\frac{1}{2}(\xi+\xi_0)^2} = n!\,(2b^2)^{-n} W_0^{\mathrm{HO}}(p,x), \tag{5.21}$$

whose substitution into (4.18) gives

$$\lim_{b\to\infty} W_n^g(p,x) = n! \lim_{b\to\infty} (2b^2)^{-n} L_n^{2A_n+1}(\rho - ib\partial_\eta) L_n^{2A_n+1}(\rho + ib\partial_\eta)\, W_0^{\mathrm{HO}}(p,x). \tag{5.22}$$

So that $\rho \pm ib\partial_\eta = \alpha + \sqrt{2\alpha}\left(\xi \pm \frac{i}{2}\partial_\eta\right)$, where $\alpha = 2b^2$, according to (4.9) will have

$$\lim_{b\to\infty}(2b^2)^{-n} L_n^{2A_n+1}(\rho - ib\partial_\eta) L_n^{2A_n+1}(\rho + ib\partial_\eta) = \\ = \frac{1}{2^n (n!)^2} H_n\left(\xi+\xi_0 - \frac{i}{2}\partial_\eta\right) H_n\left(\xi+\xi_0 + \frac{i}{2}\partial_\eta\right). \tag{5.23}$$

Taking into account (5.22) and (5.23) we find

$$\lim_{b\to\infty} W_n^g(p,x) = \frac{1}{\pi\hbar 2^n n!} H_n\left(\xi+\xi_0 - \frac{i}{2}\partial_\eta\right) H_n\left(\xi+\xi_0 + \frac{i}{2}\partial_\eta\right) e^{-\eta^2-(\xi+\xi_0)^2}, \tag{5.24}$$

i.e. at $b \to \infty$ the Wigner function of the model of a linear harmonic oscillator with position dependent mass and frequency in an external uniform field transforms into the Wigner function for a linear harmonic oscillator with constant mass and frequency in an external uniform field.

In this work, we have found an exact expression for the Wigner function of a quantum linear harmonic oscillator with position dependent mass and frequency in an external uniform field in the case of a semiconfined quantum parabolic well. Although the wave function of the considered system is defined on the half-line$(-a;\infty)$, integration in the definition of the Wigner function is carried out in the finite region $(-(x+a); x+a)$.

---

[1] *E.P. Wigner*. On the quantum correction for thermodynamic equilibrium, Phys.Rev., 40 749 1932.

[2] *M. Hillery, R.F. O'Connell, M.O. Scully and E.P. Wigner*. Distribution functions in physics: Fundamentals, Phys. Rep., 106 121, 1984.

[3] *H.-W. Lee.* Theory and application of the quantum phase-space distribution functions, Phys. Rep. 259, 1995, 147-211.

[4] *V.I. Tatarskiˇı.* The Wigner representation of quantum mechanics, Sov. Phys. Usp., 26 311, 1983.

[5] *K. Husimi*, Some formal properties of the density matrix, Proc. Phys.-Math. Soc. Japan, 22 264, 1940.

[6] *H. Weyl.* "Quantenmechanik und Gruppentheorie" Z. Phys 46 (1927) 1-33.

[7] *R.W. Davies and K.T.R. Davies.* On the Wigner distribution function for an oscillator,Ann.Phys. (N.Y.), 89 261-273, 1975.

[8] *N.M. Atakishiyev, Sh.M. Nagiyev and K.B. Wolf.* Wigner distribution functions for a relativistic linear oscillator, Theor.Math,Phys., 114 322 (1998).

[9] *E. I. Jafarov, S. Lievens, S. M. Nagiyev, J. Van der Jeught.* The Wigner functions of q-deformed harmonic oscillator model, J.Phys. A: Math. Theor., 40 5427-5441, 2007.

[10] *E. I. Jafarov, S. Lievens and J. Van der Jeught.* The Wigner distribution functions for the one-dimensional parabose oscillator, J.Phys. Theor., 41 235301, 2008.

[11] *S.M. Nagiyev, G.H. Guliyeva and E.I. Jafarov.* The Wigner functions of the relativistic finite-difference oscillator in an external field, J.Phys. A: Math. Theor., 42 454015, 2009.

[12] *K. Li, J. Wang, S. Dulat and K. Ma.* Wigner functions for Klein-Gordon oscillators in non-commutative space, Int.J.Theor.,Phys., 49 134-143, 2010.

[13] *M. Kai, W. Jian-Hua and Y. Yi.* Wigner function for the Dirac oscillator in spinor space, Chinese Phys.C, 35 11-15, 2011.

[14] *J. Van der Jeugt.* A Wigner distribution function for finite oscillator systems, J.Phys. A: Math. Theor., 46 475302, 2013.

[15] *S. Hassanabadi and M. Ghominejad.* Wigner function for Klein-Gordon oscillator in commutative and non-commutative space, Eur. Phys. J. Plus, 131 212, 2016.

[16] *Z.-d. Chen and G. Chen.* Wigner function of the position-dependent effective Schrodinger equation, Phys. Scr., 73 354-358, 2006.

[17] *A. de Souza Dutra and J.A. de Oliveira.* Wigner distribution for a class of isospectral position-dependent mass systems, Phys. Scr., 78 035009, 2008.

[18] *E.I. Jafarov, S.M. Nagiyev, R. Oste, J. Van der Jeugt.* "Exact solution of the position-dependent effective mass and angular frequency Schr¨odinger equation: harmonic oscillator model with quantized confinement parameter", J. Phys. A: Math. Theor., 53:48, 2020, 485301, 14 pp.
[19] *D.J. BenDaniel, C.B. Duke.* "Space-charge effects on electron tunneling", Phys. Rev., 152:2, 1966, 683–692.
[20] *O. Von Roos*. "Position-dependent effective masses in semiconductor theory", Phys. Rev. B, 27:12, 1983, 7547–7552.
[21] *J.-M. L´evy-Leblond.* "Position-dependent effective mass and Galilean invariance", Phys. Rev. A, 52:3, 1995, 1845–1849.
[22] *G. Bastard.* Wave Mechanics Applied to Semiconductor Heterostructure, Les Edition de Physique, Paris, 1988.
[23] *P. Harrison.* Quantum Wells, Wires and Dots: Theoretical and Computational Physics, John Wiley and Sons, New York, 2000.
[24] *M. Barranco, M. Pi, S. M. Gatica, E.S. Hern´andez, J. Navarro.* "Structure and energetics of mixed ^{4}He–^{3}He drops", Phys. Rev. B, 56:14, 1997, 8997–9003.
[25] *F. Arias de Saavedra, J. Boronat, A. Polls, A. Fabrocini.* "Effective mass of one ^{4}He atom in liquid ^{3}He", Phys. Rev. B, 50:6, 1994, 4248–4251.
[26] *T. Gora, F. Williams.* "Theory of electronic states and transport in graded mixed semiconductors", Phys. Rev., 177:3, 1969, 1179–1182.
[27] *Q.-G. Zhu, H. Kroemer.* "Interface connection rules for effective-mass wave functions at an abrupt heterojunction between two different semiconductors", Phys. Rev. B, 27:6, 1983, 3519–3527.
[28] *H. Raibongshi, N. N. Singh.* "Construction of exactly solvable potentials in the D-dimensional Schrödinger equation with coordinate-dependent mass using the transformation method", TMF, 183:2, 2015, 312–328.
[29] *N. Amir, S. Iqbal.* "Algebraic solutions of shape-invariant position-dependent effective mass systems", J. Math. Phys., 57:6, 2016, 062105.
[30] *B. Roy.* "Lie algebraic approach to singular oscillator with a position-dependent mass", Europhys. Lett., 72:1, 2005, 1–6.
[31] *J. Yu, S.-H. Dong.* "Exactly solvable potentials for the Schr¨odinger equation with spatially dependent mass", Phys. Lett. A, 325, 2004, 194–198.
[32] *J.R.F. Lima, M. Vieira, C. Furtado, F. Moraes, C. Filgueiras.* "Yet another position-dependent mass quantum model", J. Math. Phys., 53:7, 2012, 072101.
[33] *C. Quesne, V. M. Tkachuk.* "Deformed algebras, position-dependent effective masses and curved spaces: an exactly solvable Coulomb problem", J. Phys. A: Math. Gen., 37:14, 2004, 4267–4281, arXiv: math-ph/0403047.
[34] *J. F. Cari˜nena, M. F. Ra˜nada, M. Santander.* "Quantization of Hamiltonian systems with a position dependent mass: Killing vector fields and Noether momenta approach", J. Phys. A: Math. Theor., 50:46, 2017, 465202, 20 pp.
[35] *E. I. Jafarov, S. M. Nagiyev, A. M. Jafarova.* "Quantum-mechanical explicit solution for the confined harmonic oscillator model with the von Roos kinetic energy operator", Rep. Math. Phys., 86:1, 2020, 25–37.
[36] *E. I. Jafarov, Sh. M. Nagiyev.* "Angular part of the Schrödinger equation for the potential Oto as a harmonic oscillator with coordinate-dependent mass in a homogeneous gravitational field", TMF, 207:1, 2021, 58–71.
[37] *A. de Souza Dutra, A. de Oliveira.* "Two-dimensional position-dependent massive particles in the presence of magnetic fields", J. Phys. A: Math. Theor., 42:2, 2009, 025304, 13 pp.
[38] *Shakir M. Nagiyev, Shovqiyya A. Amirova,* Model of a linear harmonic oscillator with a position-dependent mass in the external homogeneous field. The case of a parabolic well, AJP Physics 28:4, 2022 section En, 36-41.
[39] *S.M. Nagiyev.* On two direct limits relating pseudo-Jacobi polynomials to Hermite polynomials and the pseudo-Jacobi oscillator in a homogeneous gravitational field. Theor. Math. Phys. 210, 2022, 121.
[40] *S.M. Nagiyev, C. Aydin, A.I. Ahmadov, S.A. Amirova.* "Exactly solvable model of the linear harmonic oscillator with a positiondependent mass under external homogeneous gravitational field", Eur. Phys. J. Plus., 540: 137, 2022,13.
[41] *A.P. Prudnikov, Yu.A. Brychkov, O. I. Marichev,* "Integrals and series", vol. 2: Special functions, Nauka, M., 1983.
[42] *H. Bateman and A. Erd´elyi.* "Higher Transcendental Functions": 2 (McGraw Hill Publications, New York, 1953).
[43] *H. Bateman and A. Erd´elyi.* "Higher Transcendental Functions": 1 (McGraw Hill Publications, New York, 1953).
[44] *R. Koekoek, P.A. Lesky and R.F. Swarttouw.* "Hypergeometric Orthogonal Polynomials and their q-Analogues", (Springer, Berlin 2010).

Received: 27.02.2023