TWO NEW LIMIT TRANSITIONS FROM JACOBI POLYNOMIALS TO HERMITE POLYNOMIALS

SHAKIR M. NAGIYEV
Institute of Physics, Ministry of Science and Education, H. Javid Avenue, 131, AZ 1073
Baku, Azerbaijan
shakir.m.nagiyev@gmail.com

We present two new limit relations that reduce the orthogonal Jacobi polynomials directly to the Hermite polynomials with shifted and nonshifted arguments. The proofs of these limit relations are based on the method of mathematical induction. The obtained limits open the way to studying new exactly solvable harmonic oscillator models in quantum mechanics in terms of Jacobi polynomials.

Keywords: Jacobi polynomials, Hermite polynomials, limit relations.
DOI:10.70784/azip.1.2025303

1. INTRODUCTION

Special functions of mathematical physics usually arise when solving differential or finite-difference equations describing mathematical models of physical objects [1–4]. These include, for example, orthogonal polynomials of continuous and discrete arguments, gamma functions, spherical, cylindrical and hypergeometric functions and their q-deformed analogues [4], etc. Special functions are widely used in all areas of theoretical and mathematical physics when considering a wide class of phenomena in atomic and molecular physics, electromagnetic radiation, optics, solid-state physics, nuclear physics and elementary particle physics, group representation theory, combinatorics, coding theory, probability theory, and computational mathematics.

It is worth noting that the hypergeometric orthogonal polynomials and their q-analogues [1–4] form an important class of special functions. This class includes various families of polynomials that form the Askey scheme or table of orthogonal polynomials introduced in the 1980s [4]. In this scheme, polynomials are arranged at different five levels depending on the number of their parameters. For example, the Wilson polynomials form a family of orthogonal polynomials with four parameters and are therefore arranged at the upper level ${}_4F_3(4)$. The Hermite polynomials do not contain parameters, so they are arranged at the lower level ${}_2F_0(0)$. The Hermite, Laguerre, Jacobi and Meixner-Pollachek polynomials, as exact solutions of second-order differential equations of hypergeometric type, or finite-difference equations, are among the most widely used in problems on bound states of relativistic and non-relativistic quantum mechanics of classical orthogonal polynomials [5–13]. This is primarily due to the fact that the exact solutions of the Schrödinger, Klein-Gordon, Dirac wave equations [5 – 8], as well as the relativistic finite-difference equation [9–13], as usual, consist of a weighted and a polynomial part, i.e. are expressed through orthogonal polynomials.

The Askey scheme reflects the "hidden" properties of orthogonal polynomials. In particular, an important aspect of the Askey scheme is that it is possible to establish relations between almost all polynomials located in the nearest or almost nearest neighboring nodes of the table by means of exact limit relations or special cases. In this regard, we emphasize that in the literature limit relations between the Meixner polynomials $M_n(x;a,c)$ and Hermite polynomials have been obtained [14]

$$\lim_{\nu\to\infty}(2\nu)^{n/2}\,M_n\left(\frac{\nu+\sqrt{2\nu}x}{1-\gamma};\frac{\nu}{\gamma},\gamma\right)=(-1)^nH_n(x), \tag{1.1}$$

between continuous dual Hahn polynomials $S_n(x^2;a,b,c)$ and Laguerre polynomials $L_n^{\alpha}(x)$ [11]

$$\lim_{\nu\to\infty}\frac{1}{n!\nu}S_n\left(x\nu;a,b,\frac{1}{2}\right)=L_n^{a_0-1/2}(x), \tag{1.2}$$

where $a_0=\lim_{\nu\to\infty}a,\quad \lim_{\nu\to\infty}(b-\nu)=\text{const},$ between Meixner–Pollaczek polynomials $P_n^{\nu}(x;\varphi)$ and Hermite polynomials [13]

$$\lim_{\nu\to\infty}n!\,\nu^{-n/2}\,P_n^{\nu}\left(x\sqrt{\nu};\arccos\frac{x_0}{\sqrt{\nu}}\right)=H_n(x+x_0), \tag{1.3}$$

between Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ with domain $-1<x<1$ and Hermite polynomials [15]

$$\lim_{\nu\to\infty} 2^n n!\,\nu^{-n}\, P_n^{(\nu^2+a\nu,\nu^2+b\nu\ )}\left(\frac{x}{\nu}\right) = H_n\left(x+\frac{a-b}{2}\right), \tag{1.4}$$

between Bessel polynomials $y_n(x;a)$ and Hermite polynomials [16]

$$\lim_{\nu\to\infty} (-1)^n (2\nu)^{n/2}\, y_n(\frac{2}{\nu}+\frac{2}{\nu}\sqrt{\frac{2}{\nu}}x;-\nu)\left(\frac{x}{\nu}\right) = H_n(x), \tag{1.5}$$

between pseudo-Jacobi polynomials $P_n(x;\nu,N)$ and Hermite polynomials [17] (see also [18])

$$\lim_{N\to\infty} 2^n N^{n/2}\, P_n\left(\frac{x}{\sqrt{N}};\nu,N\right) = H_n(x), \tag{1.6}$$

$$\lim_{N\to\infty} 2^n N^{n/2}\, P_n\left(\frac{x}{\sqrt{N}};\nu\sqrt{N},N\right) = H_n(x-\nu). \tag{1.7}$$

In addition, in [11] it is also shown that the Meixner–Pollaczek polynomials are special cases of continuous dual Hahn polynomials

$$P_{2n+1}^{b}\left(x;\frac{\pi}{2}\right) = (-1)^n \frac{2^{2n+1}}{(2n+1)!} xS_n\left(x^2;1,b,\frac{1}{2}\right), \tag{1.8a}$$

$$P_{2n}^{b}\left(x;\frac{\pi}{2}\right) = (-1)^n \frac{2^{2n}}{(2n)!} S_n\left(x^2;0,b\frac{1}{2}\right). \tag{1.8b}$$

The aim of this paper is to prove two theorems (Theorems 1 and 2 below) that in fact there are two additional direct limit transitions from Jacobi polynomials $P_n^{(\alpha,\beta)}(-x)$ with domain $1<x<\infty$, to Hermite polynomials with unshifted and shifted arguments.

2. BASIC PROPERTIES OF HERMITE AND JACOBI POLYNOMIALS

In this section we give the basic formulas for the Hermite and Jacobi polynomials. These formulas are well known [3, 4]. However, we give them here for future reference.

The Hermite polynomials are defined in terms of the ${}_2F_0$ hypergeometric functions as follows

$$H_n(x) = (2x)^n\, {}_2F_0\left(-\frac{n}{2},-\frac{n-1}{2};-\frac{1}{x^2}\right). \tag{2.1}$$

They are exact solutions of the following second-order ordinary differential equation

$$y''(x) - 2xy'(x) + 2ny(x) = 0,\ y(x) = H_n(x). \tag{2.2}$$

The orthogonality condition for Hermite polynomials is

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x)H_n(x)dx = d_n^2\delta_{mn},\ d_n^2 = 2^n n!\sqrt{\pi}. \tag{2.3}$$

The recurrence relation for them is determined by the formula

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \tag{2.4}$$

Two-parameter Jacobi polynomials belong to a higher level in the Askey scheme, and are defined in terms of ${}_2F_1$ hypergeometric functions as follows:

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\ {}_2F_1\left(\begin{matrix}-n,n+\alpha+\beta+1\\ \alpha+1\end{matrix};\ \frac{1-x}{2}\right), \tag{2.5}$$

They satisfy the following second-order ordinary differential equation

$$(1-x^2)y''(x) + [\beta-\alpha-(\alpha+\beta+2)x]y'(x) + n(n+\alpha+\beta+1)y(x) = 0, \tag{2.6}$$

where $y(x) = P_n^{(\alpha,\beta)}(x)$.

For $\beta > -1, \alpha + \beta < -2N - 1$ and $m, n \in \{0,1,2, \dots, N\}$ the orthogonality condition for Jacobi polynomials is written as

$$\int_1^{\infty} (x+1)^{\alpha} \, (x-1)^{\beta} \, P_m^{(\alpha,\beta)}(-x) \, P_n^{(\alpha,\beta)}(-x) dx = d_{Nn}^2 \delta_{mn}, \tag{2.7}$$

where

$$d_{Nn}^2(\alpha,\beta) = -\frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(-n-\alpha-\beta)\Gamma(n+\beta+1)}{\Gamma(-n-\alpha)n!}. \tag{2.8}$$

(Note. It should be noted that in formula (9.8.3) of the book [4] (p. 217) there is a typo: instead of $\Gamma(n+\alpha+\beta+1)$ it should be $\Gamma(n+\beta+1)$).

The recurrence formula for these polynomials has the form

$$P_{n+1}^{(\alpha,\beta)}(x) = A_n P_n^{(\alpha,\beta)}(x) - B_n P_{n-1}^{(\alpha,\beta)}(x), \tag{2.9}$$

$$A_n(x;\alpha,\beta) = \left[\frac{\beta^2 - \alpha^2}{2n+\alpha+\beta} - x(2n+\alpha+\beta+2)\right] \frac{2n+\alpha+\beta+1}{2(n+1)(n+\alpha+\beta+1)},$$

$$B_n(\alpha,\beta) = \frac{(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}. \tag{2.10}$$

3. Two Limit Relations between Jacobi Polynomials and Hermite Polynomials

Theorem 1 (Limit relation from $P_n^{(\alpha,\beta)}(-x)$ to $H_n(x)$). The Hermite polynomials $H_n(x)$ are obtained from the Jacobi polynomials $P_n^{(\alpha,\beta)}(-x)$, where $x \in (1;\infty)$, if we make the substitutions $\alpha \to -2\nu$, $\beta \to \nu$, $x \to 1 + 2e^{2x/\sqrt{\nu}}$, and then pass to the limit $\nu \to \infty$, in accordance with the following relation

$$\lim_{\nu\to\infty} n! \, \nu^{-\frac{n}{2}} \, P_n^{(-2\nu,\nu)} \left(-1 - 2e^{2x/\sqrt{\nu}}\right) = H_n(x). \tag{3.1}$$

Proof. We will prove relation (3.1) by mathematical induction. To do this, we first write down explicitly the Jacobi polynomials and the Hermite polynomials for the first few values of: $n = 0, 1, 2$:

$$P_0^{(\alpha,\beta)}(z) = 1, \; P_1^{(\alpha,\beta)}(z) = \tfrac{1}{2}[(\alpha+\beta+2)z + \alpha - \beta],$$

$$P_1^{(\alpha,\beta)}(z) = \tfrac{1}{8}[(\alpha+\beta+3)(\alpha+\beta+4)z^2 + 2(\alpha+\beta+3)(\alpha-\beta)z + (\alpha+1)(\alpha+2) +$$

$$+(\beta+1)(\beta+2) - 2(\alpha+2)(\beta+2)], \tag{3.2}$$

$$H_0(z_1) = 1, \; H_1(z_1) = 2z_1, \; H_2(z_1) = 4z_1^2 - 2.$$

Using these expressions, we directly obtain that for $n = 1$ and $n = 2$ the relation (3.1) is true:

$$\lim_{\nu\to\infty} \nu^{-\frac{1}{2}} \, P_1^{(-2\nu,\nu)} \left(-1 - 2e^{2x/\sqrt{\nu}}\right) = 2x = H_1(x),$$

$$\lim_{\nu\to\infty} 2\nu^{-1} \, P_2^{(-2\nu,\nu)} \left(-1 - 2e^{2x/\sqrt{\nu}}\right) = 4x^2 - 2 = H_2(\,x). \tag{3.3}$$

Now we prove that relation (3.1), valid in the cases $n = 1$ and $n = 2$, also holds for an arbitrary $n > 2$. To do this, assume that equality (3.1) holds for the polynomials $P_n^{(-2\nu,\nu)}\left(-1 - 2e^{2x/\sqrt{\nu}}\right)$ and $P_{n-1}^{(-2\nu,\nu)}\left(-1 - 2e^{2x/\sqrt{\nu}}\right)$ for some n. Then it also holds for $P_{n+1}^{(-2\nu,\nu)}\left(-1 - 2e^{2x/\sqrt{\nu}}\right)$. Indeed, we multiply both sides of the recurrence relation (3.9) for the Jacobi polynomials by $(n+1)! \, \nu^{-(n+1)/2}$. Then we pass to the limit $\nu \to \infty$ and from here we obtain recurrence relations for the Hermite polynomials (2.4). Thus, we have

$$\lim_{\nu\to\infty}(n+1)!\,\nu^{-\frac{n+1}{2}}P_{n+1}^{(-2\nu,\nu)}\left(-1-2e^{2x/\sqrt{\nu}}\right)=\bar{A}(x)H_n(x)-\bar{B}_nH_{n-1}(x). \tag{3.4}$$

Because

$$\bar{A}(x)=\lim_{\nu\to\infty}(n+1)\nu^{-\frac{1}{2}}A_n\left(-1-2e^{2x/\sqrt{\nu}};-2\nu,\nu\right)=2x,$$

$$\bar{B}_n=\lim_{\nu\to\infty}n(n+1)\nu^{-1}B_n(-2\nu,\nu)=2n, \tag{3.5}$$

then we get

$$\lim_{\nu\to\infty}(n+1)!\,\nu^{-\frac{n+1}{2}}P_{n+1}^{(-2\nu,\nu)}\left(-1-2e^{2x/\sqrt{\nu}}\right)=2xH_n(x)-2nH_{n-1}(x). \tag{3.6}$$

Therefore, taking into account (2.4), we have the equality

$$\lim_{\nu\to\infty}(n+1)!\,\nu^{-\frac{n+1}{2}}P_{n+1}^{(-2\nu,\nu)}\left(-1-2e^{2x/\sqrt{\nu}}\right)=H_{n+1}(x). \tag{3.7}$$

This completes the proof of Theorem 1.

Note that in the limit $\nu\to\infty$ the orthogonality relation (2.7) for the Jacobi polynomials $P_n^{(-2\nu,\nu)}\left(-1-2e^{2x/\sqrt{\nu}}\right)$ goes over to the orthogonality relation for the Hermite polynomials $H_n(x)$ (2.3). To prove this, we multiply both sides of (2.7) by $(n!)^2\nu^{-n}$, make the change of integration variable $x\to 1+2e^{2x/\sqrt{\nu}}$ and take into account that in this limit the weight function of the Jacobi polynomials $P_n^{(-2\nu,\nu)}\left(-1-2e^{2x/\sqrt{\nu}}\right)$ asymptotically behaves as

$$\left(2+2e^{2x/\sqrt{\nu}}\right)^{-2\nu}\left(2e^{2x/\sqrt{\nu}}\right)^{-\nu}\cong 2^{-3\nu}e^{-x^2}. \tag{3.8}$$

It is also easy to verify that in this limit the square of the norm $d_{Nn}^2(-2\nu,\nu)$ of the Jacobi polynomials $P_n^{(-2\nu,\nu)}\left(-1-2e^{2x/\sqrt{\nu}}\right)$ is asymptotically equal to the square of the norm d_n^2 of the Hermite polynomials

$$d_{Nn}^2(-2\nu,\nu)\cong 2^{-3\nu+2}\nu^{n-1/2}(n!)^{-2}d_n^2. \tag{3.9}$$

Let us also take into account the asymptotics of the integration differential

$$d(-1-2e^{2x/\sqrt{\nu}})\cong 4\nu^{-1/2}dx. \tag{3.10}$$

Now, using the limit relation (3.1) and the asymptotic expressions (3.8) – (3.10), it is easy to verify that the orthogonality condition for the Jacobi polynomials $P_n^{(-2\nu,\nu)}\left(-1-2e^{2x/\sqrt{\nu}}\right)$ does indeed transform into the orthogonality condition for the Hermite polynomials $H_n(x)$.

Note that to find the asymptotic expressions (3.8) – (3.10), we used the following asymptotic formulas, valid for $|z|\to\infty$ and $|x|\ll 1$:

$$\Gamma(z+1)\cong\sqrt{2\pi z}e^{z\ln z-z},\ \ \ln(1+x)\cong x-\tfrac{1}{2}x^2. \tag{3.11}$$

For example, the Γ-functions included in $d_{Nn}^2(-2\nu,\nu)$ have the following asymptotics:

$$\begin{gathered}\Gamma(\nu-n)\cong\sqrt{2\pi}\nu^{\nu-n-1/2}e^{-\nu},\\ \Gamma(\nu+n+1)\cong\sqrt{2\pi}\nu^{\nu+n+1/2}e^{-\nu},\\ \Gamma(2\nu-n)\cong\sqrt{2\pi}(2\nu)^{2\nu-n-1/2}e^{-2\nu}.\end{gathered} \tag{3.12}$$

In the indicated redistribution $\nu\to\infty$, equation (2.6) for Jacobi polynomials also transforms into equation (2.2) for Hermite polynomials. Indeed, if in equation (2.6) we replace x with $-1-2e^{2x/\sqrt{\nu}}$, then we obtain an equation that, after simple transformations and passage to the limit $\nu\to\infty$, coincides with equation (2.2).

Theorem 2 (Limit relation from $P_n^{(\alpha,\beta)}(-x)$ to $H_n(x+x_0)$). The Hermite polynomials $H_n(x+x_0)$ are obtained from the Jacobi polynomials $P_n^{(\alpha,\beta)}(-x)$, where $x \in (1;\infty)$, if we make the substitutions $\alpha \to -2\nu$, $\beta \to \nu - x_0\sqrt{\nu}$, $x \to 1 + 2e^{2x/\sqrt{\nu}}$, and then pass to the limit $\nu \to \infty$, in accordance with the following relation

$$\lim_{\nu\to\infty} n!\,\nu^{-\frac{n}{2}}\,P_n^{\left(-2\nu,\nu-x_0\sqrt{\nu}\right)}\left(-1-2e^{2x/\sqrt{\nu}}\right) = H_n(x+x_0). \tag{3.13}$$

Proof. We prove relation (3.13) by the method of mathematical induction. Using the explicit form (3.2) of Jacobi polynomials, it is easy to verify directly that for $n = 1$ and $n = 2$ relation (3.13) is true:

$$\lim_{\nu\to\infty} \nu^{-\frac{1}{2}}\,P_1^{\left(-2\nu,\nu-x_0\sqrt{\nu}\right)}\left(-1-2e^{2x/\sqrt{\nu}}\right) = 2(x+x_0) = H_1(x+x_0),$$

$$\lim_{\nu\to\infty} 2\nu^{-1}\,P_2^{\left(-2\nu,\nu-x_0\sqrt{\nu}\right)}\left(-1-2e^{2x/\sqrt{\nu}}\right) = 4(x+x_0)^2 - 2 = H_2(x+x_0). \tag{3.14}$$

Now we will prove that relation (3.13), which is valid in the cases $n = 1$ and $n = 2$, also holds for an arbitrary $n > 2$. To do this, assume that equality (3.13) holds for the polynomials $P_n^{\left(-2\nu,\nu-x_0\sqrt{\nu}\right)}\left(-1-2e^{2x/\sqrt{\nu}}\right)$ and $P_{n-1}^{\left(-2\nu,\nu-x_0\sqrt{\nu}\right)}\left(-1-2e^{2x/\sqrt{\nu}}\right)$ for some n. Then it also holds for $P_{n+1}^{\left(-2\nu,\nu-x_0\sqrt{\nu}\right)}\left(-1-2e^{2x/\sqrt{\nu}}\right)$. Indeed, if we multiply both sides of the recurrence relation (2.9) for the Jacobi polynomials $P_{n+1}^{\left(-2\nu,\nu-x_0\sqrt{\nu}\right)}\left(-1-2e^{2x/\sqrt{\nu}}\right)$ by $(n+1)!\,\nu^{-(n+1)/2}$ and then take the limit $\nu \to \infty$, we obtain the recurrence relations (2.4) for the Hermite polynomials $H_n(x+x_0)$, with a shifted argument:

$$\begin{aligned}\lim_{\nu\to\infty}(n+1)!\,\nu^{-\frac{n+1}{2}}\,P_{n+1}^{\left(-2\nu,\nu-x_0\sqrt{\nu}\right)}\left(-1-2e^{2x/\sqrt{\nu}}\right) &= \\ = \bar{A}(x+x_0)H_n(x+x_0) - \bar{B}_n H_{n-1}(x+x_0)&.\end{aligned} \tag{3.15}$$

Here

$$\bar{A}(x+x_0) = \lim_{\nu\to\infty} n(n+1)\nu^{-\frac{1}{2}}A_n\left(-1-2e^{2x/\sqrt{\nu}}; -2\nu, \nu - x_0\sqrt{\nu}\right) = 2(x+x_0),$$

$$\bar{B}_n = \lim_{\nu\to\infty} n(n+1)\nu^{-1}B_n\left(-2\nu, \nu - x_0\sqrt{\nu}\right) = 2n. \tag{3.16}$$

Thus,

$$\begin{aligned}\lim_{\nu\to\infty}(n+1)!\,\nu^{-\frac{n+1}{2}}\,P_{n+1}^{\left(-2\nu,\nu-x_0\sqrt{\nu}\right)}\left(-1-2e^{2x/\sqrt{\nu}}\right) &= \\ = 2(x+x_0)H_n(x+x_0) - 2nH_{n-1}(x+x_0)&.\end{aligned} \tag{3.17}$$

Therefore, taking into account (2.4), we have the equality

$$\lim_{\nu\to\infty}(n+1)!\,\nu^{-\frac{n+1}{2}}\,P_{n+1}^{\left(-2\nu,\nu-x_0\sqrt{\nu}\right)}\left(-1-2e^{2x/\sqrt{\nu}}\right) = H_{n+1}(x+x_0). \tag{3.18}$$

This completes the proof of Theorem 2.

It is easy to show that in the limit $\nu \to \infty$ equation (2.6) for the polynomials $P_n^{\left(-2\nu,\nu-x_0\sqrt{\nu}\right)}\left(-1-2e^{2x/\sqrt{\nu}}\right)$ goes over to equation (2.2) for the Hermite polynomials $H_n(x+x_0)$ with a shifted argument.

It is also easy to prove that in this limit the orthogonality relation (2.7) for the Jacobi polynomials $P_n^{\left(-2\nu,\nu-x_0\sqrt{\nu}\right)}\left(-1-2e^{2x/\sqrt{\nu}}\right)$ again goes over to the orthogonality relation (2.3) for the Hermite polynomials $H_n(x+x_0)$ with a shifted argument.

It should be noted that relations (3.1) and (3.13) open the way to the construction of new exactly solvable quantum-mechanical models of the harmonic oscillator.

4. CONCLUSION

In the presented article, using the method of mathematical induction, we proved two limit relations by which Jacobi orthogonal polynomials are transformed into Hermite polynomials. These relations can find wide application in problems of theoretical and mathematical physics, especially in quantum mechanics, and in the theory of orthogonal polynomials. For example, on their basis, it is possible to construct an oscillatory model in an external homogeneous field.

[1] *A.F. Nikiforov and V.B. Uvarov*. Special Functions of Mathematical Physics, Birkhauser, Boston, MA 1988.

[2] *A.F. Nikiforov and V.B. Uvarov*. Classical Orthogonal Polynomials of a Discrete Variable, Springer, Berlin, Heidelberg (1991).

[3] *H. Bateman and A. Erdelyi*. Higher Transcendental Functions, Vol. 2, McGraw-Hill, New York–Toronto–London 1953.

[4] *R. Koekoek, P.A Lesky, and R.F. Swarttouw*. Hypergeometric Orthogonal Polynomials and Their q-Analogues, Springer, Berlin, Heidelberg 2010.

[5] *L.D. Landau and E.M. Lifshitz*. Course of Theoretical Physics, Vol.3:Quantum Mechanics (Non-relativistic Theory), 3rd ed., Pergamon, Oxford, New York, 1977.

[6] *A.S. Davydov*. Quantum Mechanics (Pergamon Press, 1965).

[7] *S. Flügge*. Practical Quantum Mechanics (Springer, New York, 1974).

[8] *W. Greiner*. Relativistic Quantum Mechanics, 3rd edn. (Springer, Berlin, 2000).

[9] *V.G. Kadyshevskii, R.M. Mir-Kasimov, and N.B. Skachkov*. Three-dimensional formulation of the relativistic two-body problem, Part. Nucl., 2, 635–690, 1972.

[10] *N.M. Atakishiyev, R.M. Mir-Kassimov, and Sh.M. Nagiyev*. Quasipotential models of a relativistic oscillator, Theoret. and Math. Phys., 44, 1980, 592–603.

[11] *S.M. Nagiyev, E.I.*, Jafarov and R.M. Imanov. The relativistic linear singular oscillator, J. Phys. A: Math. Gen. 36 (2003) 7813–7824.

[12] *S.M. Nagiyev, E.I. Jafarov, R.M. Imanov, L. Ho morodean*. A relativistic model of the isotropic three-dimensional singular oscillator, Phys. Lett. A 134**,** 2005, 260–266.

[13] *Sh.M. Nagiyev, R.M. Mir-Kasimov*. Relativistic linear oscillator under the action of a constant external force. Wave function and dynamical symmetry group, Theor. Math. Phys. 208 (2021) 1265 – 1276.

[14] *N.M. Atakishiyev, E.I. Jafarov, S.M. Nagiyev and K.B. Wolf*. Meixner oscillators, Rev. Mex. Fis. 44, 235–244, 1998.

[15] *E.I. Jafarov, S.M. Nagiyev*. Angular part of the Schrödinger equation for the Hautot potential as a harmonic oscillator with a coordinate-dependent mass in a uniform gravitational field, Theor. Math. Phys. 207, 447–458 (2021).

[16] *E.I. Jafarov, S.M. Nagiyev*. Onthe exactly-solvable semi-infinite quantum well of the non-rectangular step-harmonic profile, Quantum Stud.: Math. Found. (2022) 9:387–404.

[17] *Sh. M. Nagiyev*. On two direct limits relating pseudo-Jacobi polynomials to Hermite polynomials and the pseodu-Jacobi oscillator in a homogeneous gravitational field, Theor. Math. Phys. 210, 447–458 (2021).

[18] *E.I. Jafarov, Aygun M. Mammadova, Joris Van der Jeugt*. On the direct limit from pseudo-Jacobi polynomials to Hermite polynomials, Mathematics, 9, 88, 8 pp. (2021).

Received: 20.05.2025