

ASYMMETRIC SPECIAL POTENTIALS IN LOW-DIMENSIONAL SYSTEMS

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Asymmetric Special Potentials arise when spatial inversion symmetry is broken, leading to profound changes in confinement, spectra, tunneling, and transport. This review summarizes the theoretical foundations and mathematical structure of Asymmetric Special Potentials, highlights their impact on quantum wells, wires, and dots, and explores experimental realizations in semiconductor and photonic platforms. Key results—parity mixing, asymmetric tunneling, non-reciprocal propagation, and activated forbidden transitions—are highlighted. Asymmetric Special Potentials provide a unified framework for a new generation of quantum and optoelectronic devices.

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INTRODUCTION

Asymmetric Special Potentials represent a wide range of physical potentials characterized by a violation of spatial inversion symmetry. Their significance stems from the observation that actual physical systems—such as semiconductor heterostructures, cold-atom lattices, molecular vibrations, and designed metamaterials—typically do not maintain perfect symmetry. Even minor asymmetries can significantly impact spectral characteristics, tunneling rates, transport processes, optical selection rules, and thermodynamic behaviors.

The characteristics of asymmetric potentials have been thoroughly examined in numerous books and review articles. The Morse potential, which represents the asymmetric vibrations of molecular bonds, was first presented in reference [1], serving as a foundational element in Asymmetric Special Potentials research. The Rosen–Morse II potential is also introduced, accompanied by an analytical solution for the asymmetric hyperbolic potential, as discussed in [2]. This establishes the classical solvable model for Asymmetric Special Potentials. Reference [3] delves into the concept of quasi-solutions within asymmetric double-well potentials, highlighting tunnel splitting and how asymmetry influences energy levels.

The breaking of symmetry in potentials and the mixing of wave functions with respect to parity is elaborated upon in [4], which is a classic reference for analytical methodologies pertaining to Asymmetric Special Potentials. The sections addressing heterostructures and tilted quantum wells illustrate the emergence of Asymmetric Special Potentials in real materials and elucidate the effect of electric fields on band profiles, as noted in [5]. A detailed analysis of the wave function, matrix elements, and optical transitions for asymmetric quantum wells in GaAs/AlGaAs systems demonstrates one of the most practical applications of Asymmetric Special Potentials [6].

Additionally, studies on asymmetric and time-dependent potentials investigate directed tunneling

and quantum ratchet mechanisms, providing a valuable resource for understanding the dynamic features of Asymmetric Special Potentials [7]. The concept of directed transport resulting from asymmetric potentials is explained in [8], linking Asymmetric Special Potentials to statistical and nonequilibrium theories. Superfluid-insulator transitions in asymmetric double-well potentials, relevant to cold atom systems, are addressed in [9]. The effects of asymmetric optical potentials and atomic localization within optical lattices are discussed in [10], highlighting experimental applications of Asymmetric Special Potentials. Changes in spectral properties due to asymmetric confinement geometry, along with modifications of the Aharonov–Bohm effect, are presented in [11], which is useful for 2D or toroidal Asymmetric Special Potentials scenarios.

The intricacies of tilt-induced asymmetry, miniband dispersion, and variations in interband absorption in superlattices are thoroughly explained in [12], making it a key reference for macroscopic applications of Asymmetric Special Potentials. Tunneling in dissipative media and its interaction with asymmetric barriers are examined in [13], serving as a fundamental resource for understanding Asymmetric Special Potentials in dissipative contexts. Topics such as tunneling, slip, metastable states, and thermal activation in asymmetric potentials are covered in detail in [14]. Lastly, reference [15] discusses the nanotocking effects induced by asymmetric potentials, marking it as one of the significant applications of Asymmetric Special Potentials.

Asymmetric Special Potentials are not merely abstract mathematical constructs—they serve as powerful practical tools for modeling emerging nanoscale quantum devices. Advances in modern heterostructure engineering now allow precise manipulation of band-edge profiles, graded barriers, external electromagnetic fields, and compositional asymmetry. These controllable parameters generate confinement landscapes in which spatial inversion symmetry is explicitly broken, producing parity-mixed

eigenstates and enabling optical transitions that would be forbidden in symmetric systems.

The purpose of this review is to deliver a broad and exhaustive examination of Asymmetric Special Potentials, covering: (1) their mathematical definitions; (2) key analytical properties; (3) spectral and dynamical characteristics; (4) implementations in quantum wells, nanowires, and quantum dots; and (5) pathways toward realistic experimental realization. The presentation is designed to be equally useful for theoretical physicists and device-oriented engineers.

Asymmetric Special Potentials represent a fundamental category of potentials in contemporary theoretical and applied physics, distinguished by the deliberate violation of spatial inversion symmetry. In contrast to symmetric potentials satisfying $V(x)=V(-x)$ asymmetric potentials introduce spatially varying confinement, displaced equilibrium positions, mixed-parity wavefunctions, and direction-dependent physical responses. Such potentials arise naturally in a wide spectrum of low-dimensional nanostructures including:

Asymmetric Special Potentials arise in a wide variety of advanced physical systems, including [16-23].

- (a) semiconductor heterostructures featuring graded or tilted band-edge profiles,
- (b) quantum wells and superlattices subjected to strong electric or magnetic fields,
- (c) photonic metamaterials designed for directional or non-reciprocal response.

Their relevance has increased dramatically due to applications in:

- (a) quantum cascade lasers, high-efficiency infrared and terahertz photodetectors,
- (b) nanostructures with strong spin-orbit coupling, and
- (c) non-reciprocal photonic/phononic devices and asymmetric tunneling architectures.

Analytical expressions for the tunneling probability in asymmetric barriers are presented [16]. In the [17] shows how asymmetry in Josephson junctions weakens tunneling. Explains [18] the mechanism of unidirectional light propagation in asymmetric photonic potentials. In the [19] a quantum-mechanical analog of asymmetric optical potentials is presented. Describes the creation of asymmetric miniband engineering by alternating fields in semiconductor superlattices [20].

In the [21] dipole transitions and optical selection rules in asymmetric nanosystems are described in detail. In the [22] demonstrates asymmetric localization and tunneling changes in the Stark field. Describes nonlinear transport in asymmetric superlattices [23].

Asymmetric Special Potentials have a profound impact on system behavior, strongly modifying spectral characteristics, tunneling probabilities, optical transition strengths, and thermodynamic responses [24-33]. Moreover, they provide versatile model potentials for studying systems driven far from equilibrium.

This review provides a comprehensive and rigorous treatment of Asymmetric Special Potentials, covering: Theoretical background and mathematical classification

Analytical and numerical approaches to their spectra

Asymmetry-driven effects in quantum and classical dynamics Realistic models in semiconductor and photonic structures [33]

Experimental realizations and state-of-the-art fabrication techniques Case studies and open research problems.

In the [25] explains asymmetric potential changes and energy level modifications in ion traps. In the [26] explains solutions of the one-dimensional and two-dimensional nonlinear Schrödinger equation with asymmetric potentials. Presents modern results [27] on soliton deformation induced by asymmetric potentials.

In the [28] demonstrates the effect of asymmetric constraints on topological phases. Explains nonlinear oscillations and resonant structures in asymmetric potentials [29]. Introduces nonlinear asymmetric dynamics in the broad sense [30].

Classical analysis of asymmetric light traps and the behavior of directed solitons [31]. Analyzes Bloch wave functions in asymmetric periodic potentials [34]. The influence of tilt-induced asymmetry on miniband tunneling is discussed in detail [35]. Programmable asymmetric potentials in qubit-ion traps are demonstrated [36]. In the [37] electron energy and band dispersion in asymmetric crystals are explained classically. One of the most comprehensive reviews [38] of research on asymmetric double-well structures. The "symmetry breaking \rightarrow current generation" mechanism in asymmetric potentials is demonstrated [39].

The significance of Asymmetric Special Potentials continues to grow, driven by advances in nanofabrication, high-precision semiconductor growth, and the controlled application of external fields. While symmetric potentials offer elegant mathematical structure and serve as essential educational tools, real physical systems almost invariably exhibit some degree of asymmetry—either as an unavoidable feature or as a deliberately engineered property.

A thorough understanding of Asymmetric Special Potentials is therefore crucial for realistic modeling and for enabling functionalities not accessible in symmetric environments. Breaking inversion symmetry fundamentally alters energy spectra, selection rules, transport pathways, and dynamical responses. These modifications permit new modes of charge and heat transport, enhance optical transition strengths, unlock additional tunneling channels, and enable non-reciprocal behaviors.

For these reasons, Asymmetric Special Potentials are central to contemporary research and technological development in areas such as semiconductor device design, nonlinear and quantum optics, quantum information science, photonics, and architected mechanical metamaterials.

Foundations of Potential Theory

This section develops the essential concepts behind potential-energy landscapes and highlights how symmetry governs physical behavior in classical and quantum systems.

In classical mechanics, the potential energy $V(x)$ defines the force through the relation

$$F(x) = -\frac{dV}{dx}.$$

Symmetric potentials—such as the harmonic oscillator or the square well—produce predictable oscillatory motion about an equilibrium point, with the oscillation frequency determined by the curvature of the potential at that point.

Common classical potentials include:

- the parabolic (harmonic) potential,
- the square-well confinement, and
- periodic cosine-type potentials.

The Role of Symmetry. When a system satisfies $V(x)=V(-x)$, it possesses inversion symmetry. This symmetry ensures balanced forces, symmetric equilibrium positions, and, in the quantum domain, a clear classification of states into even and odd parity. Symmetry imposes strict constraints on selection rules, transition amplitudes, and overall spectral structure.

Asymmetry in a potential landscape can originate from multiple physical mechanisms, including external electric fields, compositional grading in semiconductor alloys, strain distributions, phase-shifted optical fields, and intrinsically anharmonic interactions. Even weak asymmetry leads to noticeable modifications in equilibrium positions, tunneling probabilities, and local dynamical behavior.

Consequences of Symmetry Breaking. When inversion symmetry is broken, several characteristic effects appear:

- equilibrium positions shift away from the center,
- tunneling barriers become direction-dependent,
- degeneracies inherent to symmetric systems are removed,
- transitions that were previously forbidden by parity selection rules become allowed,
- wavefunctions no longer possess definite parity.

These concepts form the foundational basis for understanding the physics of Asymmetric Special Potentials.

Definition of Asymmetric Special Potentials

Asymmetric Special Potentials encompass all potential-energy profiles in which spatial inversion symmetry is intentionally or naturally violated. In other words, an Asymmetric Special Potentials is any potential for which

$$V(x) \neq V(-x).$$

Such asymmetry can stem from material composition, engineered geometric design, externally applied electric or magnetic fields, or nonlinear interactions inherent to the system.

In contrast to symmetric potentials, Asymmetric Special Potentials produce qualitatively different phenomena: wavefunctions do not carry a definite parity, left–right tunneling amplitudes become unequal, and new optical or vibrational transitions—prohibited in symmetric environments—become activated. As a result, Asymmetric Special Potentials are vital in the physics of low-dimensional semiconductor heterostructures, nonlinear oscillators, optical lattices, and engineered metamaterial systems.

Conceptual Framework. The key conceptual idea behind Asymmetric Special Potentials is that breaking inversion symmetry introduces a directional preference into the potential landscape. This bias may be:

- **global**, such as a uniform tilt generated by an electric field, or
- **local**, such as asymmetry in the barrier height on either side of a quantum well.
- Asymmetric Special Potentials can thus be classified according to their origin:
- **Intrinsic asymmetry** arising from microscopic or molecular structure (e.g., anharmonic diatomic potentials).
- **Engineered asymmetry** created through band-structure design or compositionally graded semiconductors.
- **Field-induced asymmetry** produced when external electric or magnetic fields distort an otherwise symmetric configuration.
- These mechanisms lead to displaced equilibrium points, directional tunneling, and eigenstates with mixed parity.

Mathematical Classification.

Asymmetric Special Potentials can be organized into several mathematically meaningful categories:

- **Analytically solvable potentials**, such as the Morse and Rosen–Morse families.
- **Quasi-exactly solvable models**, including tilted double wells and asymmetric quartic potentials.
- **Numerical or simulation-derived potentials**, obtained directly from experiments, density-functional theory, or device simulations.
- Quantitative measures of asymmetry include:
- the skewness of $V(x)V(x)V(x)$,
- differences between left–right barrier heights,
- displacement of the expectation value of position $x_0 = \langle x \rangle_{x_0} = \langle x \rangle_{x_0}$.

These descriptors allow systematic characterization and comparison of Asymmetric Special Potentials models across different geometries and dimensionalities.

Physical Systems Exhibiting Asymmetry

Semiconductor Heterostructures. Semiconductor quantum wells, nanowires, and quantum dots frequently acquire asymmetry through several mechanisms, including:

- electric-field-induced tilting of conduction and valence band edges,

- compositional grading along the growth axis,
- lattice mismatch between layers and the resulting strain fields,
- asymmetric barrier heights arising from heterointerface material selection.

These factors produce shifted subband energies, stronger mixing of even- and odd-parity wavefunctions, and the emergence of optical transitions that are forbidden in symmetric structures. Asymmetric confinement is a key operating principle in devices such as quantum cascade lasers, mid-IR detectors, and terahertz photodetectors.

Photonic Metamaterials. Photonic and mechanical metamaterials are frequently built from unit cells with deliberate geometric asymmetry. Such designs enable:

- non-reciprocal propagation of electromagnetic waves,
- direction-dependent stiffness or elastic response,
- programmable or reconfigurable effective potential landscapes.

Enhanced Optical and Electronic Control. Breaking spatial symmetry provides powerful tuning capabilities:

- dipole matrix elements become adjustable over wide ranges,
- transitions can be engineered to occur at targeted energies,
- specific absorption pathways may be selectively strengthened or suppressed.

These effects underpin the operation of THz detectors, electro-optic modulators, and advanced intersubband devices.

Manipulation of Quantum States. Asymmetric confinement permits refined control over electronic and optical quantum states. By sculpting the potential landscape, one can manipulate wavefunction localization, coherence properties, and spin-dependent dynamics with high precision.

Asymmetric Resonance Engineering. Tailoring potential asymmetry also allows the design of customized resonance conditions, enabling:

- enhanced mode confinement,
- high-Q resonances,
- selective coupling in photonic and phononic systems.

Such engineered resonances are essential in nanophotonic circuits, resonant tunneling devices, and cavity-enhanced sensors.

Mathematical Formulation of Asymmetric Special Potentials. The mathematical description of Asymmetric Special Potentials provides the foundation needed to analyze their spectral, dynamical, and transport characteristics. Symmetric

potentials typically admit closed-form solutions, parity-defined eigenstates, and simplified selection rules. In contrast, Asymmetric Special Potentials require a more general framework because parity is not conserved, analytical solutions may not exist, and energy-level structure becomes more complex.

This section develops the formal Schrödinger framework for Asymmetric Special Potentials in one-, two-, and three-dimensional geometries, focusing on the most relevant nanostructures. We will present the Hamiltonian operator, energy spectrum, and wavefunctions for asymmetric confinement in:

1. **Quantum wells (1D)**
2. **Quantum wires (2D confinement)**
3. **Quantum dots (3D confinement)**

Each system will be formulated rigorously, including external fields and asymmetry parameters.

1. ASYMMETRIC QUANTUM WELL

Quantum wells are one of the most widely studied platforms exhibiting Asymmetric Special Potentials behavior. A typical AQW consists of a thin semiconductor layer confined between two barriers, with asymmetry introduced through: different left/right barrier heights, electric-field-induced band tilting, graded alloy composition.

Wavefunction Localization

In symmetric wells, bound states exhibit definite parity. In AQWs, wavefunctions skew toward the lower potential side. The ground state localizes most strongly, while excited states show increasing delocalization.

Operator Hamiltonian

$$H = -\frac{\hbar^2}{2m^*} \frac{d^2}{dz^2} + V(z)$$

where m^* is the effective mass. Asymmetry enters directly through the functional form of $V(z)$.

a) Tilted (Electric-Field) Quantum Well. Typical examples include:

$$V(z) = V_0(z) + eFz$$

(b) Asymmetric finite well

$$V(z) = \begin{cases} V_L, & z < 0 \\ 0, & 0 < z < L \\ V_R, & z > L \end{cases} \quad V_L \neq V_R$$

Wave function
Inside the well:

$$\psi(z) = A \sin(kz) + B \cos(kz) \quad k = \sqrt{\frac{2m^*E}{\hbar^2}}$$

On the left barrier:

$$\psi_L(z) = C e^{k_L z}, \quad K_L = \sqrt{\frac{2m^*(V_L - E)}{\hbar^2}}$$

On the right barrier:

$$\psi_R(z) = D e^{-k_R(z-L)}$$

The effect of asymmetry
 Energy levels shift left/right.
 Asymmetry modifies subband energies:
 subband spacing becomes non-uniform,
 Stark shifts appear under applied electric fields,
 transition energies move to higher or lower ranges
 depending on tilt direction.

The wave function is no longer symmetric at the center → asymmetric localization.

The dipole matrix element $\langle m|z|n \rangle$ gains additional non-zero components. Dipole matrix: new links are activated due to parity violation. Transitions forbidden in symmetric systems (e.g., 1→3, 2→4) become allowed, producing extra absorption peaks observed experimentally in mid-infrared and THz spectroscopy

ASYMMETRIC QUANTUM WIRE

In a quantum wire, electron motion is free along 1D, but there is asymmetric confinement along the radial or y-axis. Quantum wires confine electrons in two dimensions, allowing free motion along one axis. Asymmetry arises through anisotropic parabolic confinement, lateral tilts, or material grading.

Anisotropic Confinement

If $\omega_x \neq \omega_y$, confinement strengths along two axes differ, creating direction-dependent localization.

$$E_{n,m} = \hbar\omega_x \left(n + \frac{1}{2} \right) + \hbar\omega_y \left(m + \frac{1}{2} \right) + \Delta E_{asym}$$

Subband Structure

In AQWRs, subbands $E_n(k)$ are highly sensitive to asymmetry. Effects include: different curvature for positive vs negative k, altered effective masses, polarization-sensitive intersubband transitions.

Transport Implications

Asymmetry leads to directional transport and non-reciprocal conductance when combined with magnetic fields or spin-orbit interactions.

ASYMMETRIC QUANTUM DOT

Quantum dots exhibit fully quantized 3D confinement. Asymmetry arises from elliptic geometries, off-center potentials, applied fields, or strain.

Hamiltonian operator

$$H = \frac{\hbar^2}{2m^*} \nabla^2 + V(x, y, z)$$

Potential:

(a) Elliptic + Tilt

$$E_{n_x, n_y} = \hbar\omega_x \left(n_x + \frac{1}{2} \right) + \hbar\omega_y \left(n_y + \frac{1}{2} \right)$$

Dipole transitions broaden: Optical absorption spectrum has two maxima (asymmetric splitting).

Spectral and Spatial Effects

Adding a tilt term αx displaces wavefunctions and changes selection rules for polarization-dependent optical absorption.

Hamiltonian operator

$$H = -\frac{\hbar^2}{2m^*} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + V(x, y)$$

Let's choose an asymmetric potential:

$$V(x, y) = \frac{1}{2} m^* (\omega_x^2 x^2 + \omega_y^2 y^2) + \alpha x$$

$\omega_x \neq \omega_y$ structural asymmetry

αx external field or internal strain

The wave function

Although not separable, in weak asymmetry:

$$\psi(x, y) = \phi_n(x) \chi_m(y)$$

Here:

$$\phi_n(x) = H_n(\xi) e^{-\xi^2/2} \quad \xi = \sqrt{\frac{m^* \omega_x}{\hbar}} x$$

but the tilt term shifts the wave function to the left or right.

$$x_0 = -\frac{\alpha}{m^* \omega_x^2}$$

Energy levels differ in two directions:

$$V(x, y) = \frac{1}{2} m^* (\omega_x^2 x^2 + \omega_y^2 y^2) + \alpha x$$

(b) Asymmetric Morse-type dot

$$V(r) = D(e^{-2ar} - 2e^{-ar})$$

Wave function

A variational wavefunction, often used in the case of non-separability:

$$\psi(x, y) = \exp[-(ax^2 + by^2) + cx]$$

Here

$a \neq b$ - elliptic asymmetry

cx - tilt element

Normalization:

$$N = \sqrt{\frac{4\sqrt{ab}}{\pi}} \exp\left(\frac{c^2}{4a}\right)$$

Asymmetric Quantum Dot effects

Energy levels (s,p,d orbitals) split:

Wavefunctions lose s/p/d classification purity due to parity mixing. Level degeneracies are lifted.

The ground-state probability density shifts away from the geometric center.

Optical Signatures

Discrete absorption/emission lines shift, split, and new weak transitions appear. AQDs are widely studied in quantum information, single-photon sources, and mid- IR emitters.

Boundary Conditions and Constraints.

Asymmetry modifies the qualitative structure of boundary conditions. For symmetric potentials:

$$\psi(-x) = \pm\psi(x)$$

Nodes appear in predictable symmetric patterns

For Asymmetric Special Potentials:

Boundary amplitudes differ at $x \rightarrow -\infty$ and $x \rightarrow +\infty$ Nodes shift toward the higher-energy side

Tunneling amplitudes differ for opposite directions

This results in asymmetric scattering properties and direction-dependent transport.

Mathematical Consequences of Asymmetry.

Breaking symmetry alters several fundamental mathematical features:

Loss of Parity

Wavefunctions cannot be labeled as even or odd. This lifts degeneracies and changes the structure of optical and tunneling transitions.

Mixing of Orthogonality Classes

Eigenstates remain orthogonal, but the loss of parity removes the simple classification into two families, complicating expansions and selection rules.

Modified Ladder-Operator Structures

For harmonic-like asymmetric potentials, creation and annihilation operators acquire additional terms. The algebra no longer corresponds to the standard harmonic oscillator.

Analytical Properties of Asymmetric Special Potentials

Analytical properties of asymmetric special potentials form the core of understanding how symmetry breaking modifies the qualitative and quantitative behavior of physical systems. Here provides the mathematical and physical tools necessary to analyze stability, equilibria, perturbative responses, tunneling asymmetry, and wavefunction deformation.

Perturbation Theory in Asymmetric Potentials. For symmetric potentials perturbed by a small asymmetric term $\lambda W(x)$, first-order energy shifts are:

$$\Delta E_n^{(1)} = \lambda \langle n | W(x) | n \rangle.$$

For $W(x) = x$ (common tilt), the matrix element $\langle n|x|n \rangle$ is zero for symmetric potentials but becomes non-zero for ASP due to broken parity or the mixing of basis states. Higher-order perturbation theory contributes additional shifts and state mixing.

Parity Mixing and Wavefunction Skewness. A hallmark of ASP is the loss of definite parity. Instead of purely even or odd wavefunctions, the eigenstates acquire mixed character. Define skewness:

$$S_n = \int x |\psi_n(x)|^2 dx.$$

For symmetric potentials, $S_n = 0$. For Asymmetric Special Potentials, $S_n \neq 0$ for most states. The degree of skewness increases with asymmetry

strength and contributes to changes in transition amplitudes.

5.5 Asymmetric Selection Rules

In symmetric potentials, selection rules follow $\Delta n = \pm 1$ for harmonic systems or parity-based constraints for wells and dots. When asymmetry appears:

$\Delta n = \pm 2, \pm 3$ transitions become weakly allowed, optical absorption intensities are modified by changes in overlap integrals, dipole matrix elements gain additional non-zero components. This profoundly affects intersubband absorption spectra in quantum wells and energy-level transitions in quantum dots.

5.6 Asymmetric Tunneling and Non-Reciprocal Transport

Direction-dependent tunneling is one of the most striking features of Asymmetric Special Potentials. When the potential barrier is asymmetric, the transmission probability satisfies:

$T_{L \rightarrow R} \neq T_{R \rightarrow L}$. This leads to:

quantum ratchet transport,
diode-like tunneling behavior,
non-reciprocal wave propagation,
directional escape rates in metastable systems.

Analytical Tools for Asymmetric Special Potentials. Several analytical approaches are especially useful:

WKB approximation for tunneling asymmetry,
variational methods for estimating shifted wavefunctions,
Sturm-Liouville theory for analyzing level spacing,
perturbative expansions for weak asymmetry,
harmonic approximation near displaced equilibrium points.

These properties set the stage for understanding the spectral behavior of **Spectral Characteristics**. The spectral properties of asymmetric special potentials (Asymmetric Special Potentials) differ qualitatively from their symmetric counterparts. Breaking inversion symmetry alters the energy-level spacing, removes degeneracies, activates new optical transitions, and reshapes the density of states. These spectral features form the backbone of observable physical effects in semiconductor heterostructures, optical traps, quantum dots, and nonlinear oscillators.

Eigenvalue Problems in Asymmetric Potentials. In symmetric potentials, the Schrödinger eigenvalue problem often separates into even and odd parity sectors, simplifying both analytical and numerical solutions. For Asymmetric Special Potentials, parity is no longer a good quantum number, requiring full matrix diagonalization. The general problem is:

$$H\psi_n = E_n \psi_n,$$

with no underlying symmetry to reduce computational cost. As a result: eigenvalues are non-degenerate except in special accidental cases, eigenstates exhibit mixed parity, the ordering and spacing of levels deviate from symmetric analogs (e.g., harmonic oscillator).

Approximate analytical methods include perturbation theory, WKB quantization, and variational estimation of shifted equilibrium positions.

Even slight asymmetry produces measurable shifts in energy levels. Consider a symmetric potential $V_0(x)$ perturbed by an asymmetric term $\lambda W(x)$. To first order:

$$\Delta E_n \approx \lambda \langle n | W(x) | n \rangle.$$

In symmetric cases, $W(x) = x$ yields zero contribution due to parity. In ASP, however, eigenfunctions do not possess definite parity, so the expectation value is non-zero. Consequences include: upward or downward shifts depending on the sign of asymmetry, non-uniform modification of level spacing, enhanced splitting between adjacent excited states. This effect is especially strong in semiconductor quantum wells under applied electric fields (Stark effect).

Tunneling Splitting and Asymmetric Double Wells

In symmetric double wells, the ground and first excited states form nearly degenerate symmetric/antisymmetric pairs split by tunneling. Asymmetry lifts this degeneracy sharply. If the left and right minima differ by Δ , the tunneling-split states become localized, and the energy splitting becomes:

$$\Delta E_{\text{split}} \approx \sqrt{\Delta^2 + 4|T|^2},$$

where T is the tunnel coupling matrix element. When Δ dominates, splitting is governed primarily by asymmetry rather than tunneling.

This leads to: localization of eigenstates in the deeper well, strongly directional tunneling rates, biased superposition states important in quantum information applications.

Density of States (DOS) Modifications

Asymmetric potentials alter the local and global density of states. In low-dimensional systems, DOS is highly sensitive to confinement shape:

1D Asymmetric Special Potentials: DOS exhibits broadened or shifted van Hove singularities.

2D Asymmetric Special Potentials: Anisotropic confinement produces direction-dependent subband edges.

3D Asymmetric Special Potentials (quantum dots): discrete spectrum loses clustering symmetry.

In graded semiconductor structures, position-dependent effective masses and tilted band edges introduce additional DOS reshaping, affecting optical absorption and carrier population dynamics.

6.5 Activation of Forbidden Transitions

One of the most significant spectral effects of Asymmetric Special Potentials is the activation of transitions that are forbidden in symmetric potentials. Dipole matrix elements:

$$M_{mn} = \langle m | x | n \rangle,$$

are non-zero only for specific parity combinations in symmetric systems. In Asymmetric Special Potentials, parity mixing yields: additional absorption peaks (e.g., $1 \rightarrow 3$, $1 \rightarrow 4$), enhanced intersubband coupling, modified oscillator strengths.

This effect is prominent in asymmetric quantum wells and dots, where experimental spectra display extra lines not predicted by symmetric models.

6.6 Spectral Fingerprints of Asymmetry

Asymmetric Special Potentials introduce distinctive spectral signatures that can be used experimentally to identify asymmetry:

Shifted ground-state energy compared to symmetric models,

Uneven level spacing (anharmonicity + tilt),

Appearance of side peaks in absorption,

Asymmetric splitting patterns in quantum dots,

Directional-dependent tunneling-associated linewidths. ASP.

SP in Quantum Mechanics

Asymmetric Special Potentials play a central role in quantum mechanics because the breaking of inversion symmetry directly affects quantization, eigenstates, tunneling, and optical selection rules. Low-dimensional semiconductor structures—quantum wells, quantum wires, and quantum dots—are particularly sensitive to asymmetry. In these systems, even a small external field, barrier imbalance, or compositional gradient can lead to major modifications in electronic structure and device performance.

This section examines how Asymmetric Special Potentials influence quantum confinement, wavefunction localization, tunneling processes, and optical transitions.

7.4 Tunneling in Asymmetric Nanostructures

Asymmetric Special Potentials modify tunneling rates dramatically. The transmission probability becomes direction-dependent:

$$T_{L \rightarrow R} \neq T_{R \rightarrow L}.$$

This phenomenon is crucial in:

resonant tunneling diodes,

quantum cascade lasers,

ratchet devices.

Asymmetric tunneling produces distinct linewidth broadening and spectral asymmetry.

7.5 Role of External Fields

Electric and magnetic fields are powerful tools for tuning Asymmetric Special Potentials behavior:

Electric Fields

Introduce linear or nonlinear tilts, enabling fine control over subband energies and dipole transitions.

Magnetic Fields

Modify Landau levels in asymmetric potentials, generating: anisotropic magneto-absorption, spin splitting influenced by Rashba terms, altered cyclotron resonance.

Asymmetric Special Potentials and Spin-Orbit Interaction

Spin-orbit effects amplify asymmetry. For example, Rashba coupling $\alpha_R(\sigma \times k) \cdot z$ is proportional to structural inversion asymmetry (SIA). Combining Asymmetric Special Potentials with spin-orbit interaction results in:

spin-split subbands,

spin-dependent tunneling,

electrically tunable spin precession.

These features are essential for spintronic device engineering.

Applications

Asymmetric Special Potentials appear naturally or can be engineered deliberately in a wide variety of physical systems. Their unique ability to break inversion symmetry enables phenomena that are impossible in symmetric environments, including directional transport, non-reciprocal optical/phononic propagation, asymmetric tunneling, and highly tunable resonance behavior. This section surveys the major domains where Asymmetric Special Potentials play a central role, emphasizing realistic devices, materials platforms, and experimentally observable consequences.

Semiconductor Physics. Semiconductor nanostructures—quantum wells, wires, dots, and superlattices—provide some of the richest manifestations of Asymmetric Special Potentials.

Quantum Wells

Asymmetric quantum wells (AQWs) are formed by unequal barrier heights, compositional grading (e.g., GaAs/AlGaAs with graded Al concentration), external electric fields causing band tilting.

Applications: infrared detectors (intersubband transitions enhanced by asymmetry), quantum cascade lasers (QCLs) where strong asymmetry optimizes tunneling and lasing transitions, THz emitters using engineered parity mixing.

Quantum Wires and Nanoribbons

Lateral asymmetry ($\omega_x \neq \omega_y$) or built-in strain modifies subband dispersions, producing: polarization-dependent absorption, non-reciprocal electron transport, spin-split subbands when combined with Rashba coupling. These effects are key in spintronics, nanoscale waveguides, and ballistic transport devices.

Quantum Dots

Asymmetric quantum dots (AQDs) display: lifted degeneracies, new allowed optical transitions, strongly shifted emission spectra.

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